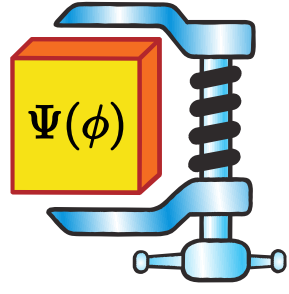
Some progress on the variational method in quantum field theory

for *Habilitation à Diriger les Recherches*

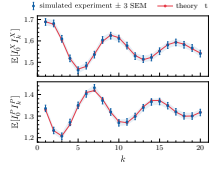


Antoine Tilloy
January 29th, 2026
Paris, France

MINES PARIS | PSL ENS | PSL Inria erc

3 lines of research – past 10 years

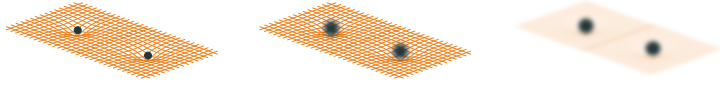
1. **How can we monitor and control few-body open quantum systems?**



→ ALICE & BOB

Pierre Guilmin [defended 2025, now Atlantic ocean], Roberto Negrin [PhD student], Florent Goulette [PhD student], Gustave Robichon [PhD student]

2. **Is gravity fundamentally quantum or classical?**

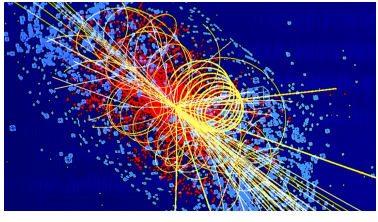


3. **Can we compute non-perturbatively in quantum field theory** ← today

Karan Tiwana [defended 2025, now postdoc], Edo Lauria [postdoc], Sophie Mutzel [postdoc], Molly Kaplan [postdoc]

Renormalizable Quantum Field Theories

♠ What can be valid at all scales → Our most *fundamental* prediction framework



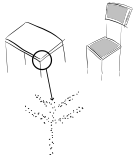
♠ What remains at large scale → A great *effective* prediction framework



The 3 difficulties of quantum field theory

1. **What does it say about the world?**
   $\rightarrow$ the measurement problem, untamed

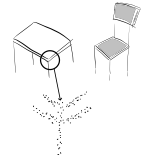




The 3 difficulties of quantum field theory

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2. **How can it be defined rigorously?**
   $\rightarrow$ Axiomatic then constructive QFT: $\phi^4_2 \rightarrow \phi^4_3 \rightarrow$??

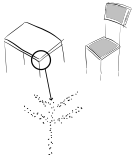




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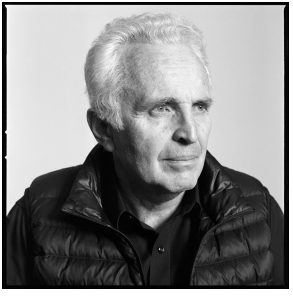


3. **How can we compute with it?** $\leftarrow$ `today`

Alexander Polyakov's dream

QCD must be exactly soluble, or else I cannot imagine what the physics textbooks of the future will look like.

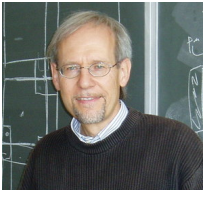
Alexander Polyakov, [quoted by John Preskill]



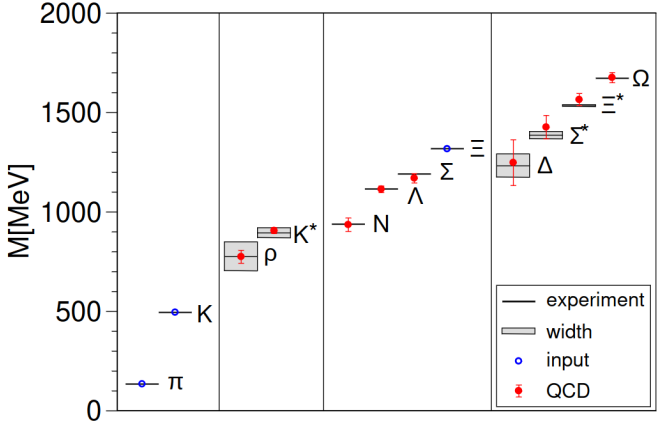
Martin Lüscher's reality

The lattice gauge theory odyssey, from toy models to QCD





Success of lattice gauge theory



Science, 2008, BMW collaboration

but a feeling of not enough...

Strategies to solve interacting QFT

Two ways to attack *real world* quantum field theories non-perturbatively

1. Start **simpler** so that it becomes **simpler** [*e.g.* ϕ_2^4]
2. Start **more complex** so that it becomes **simpler** [*e.g.* $\mathcal{N} = 4$ *SYM*]



ϕ_2^4 - pile of dirt

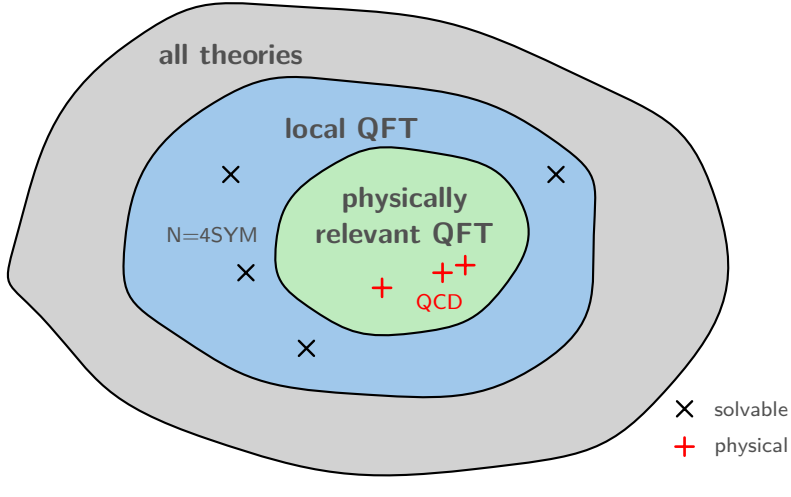


QCD - Everest

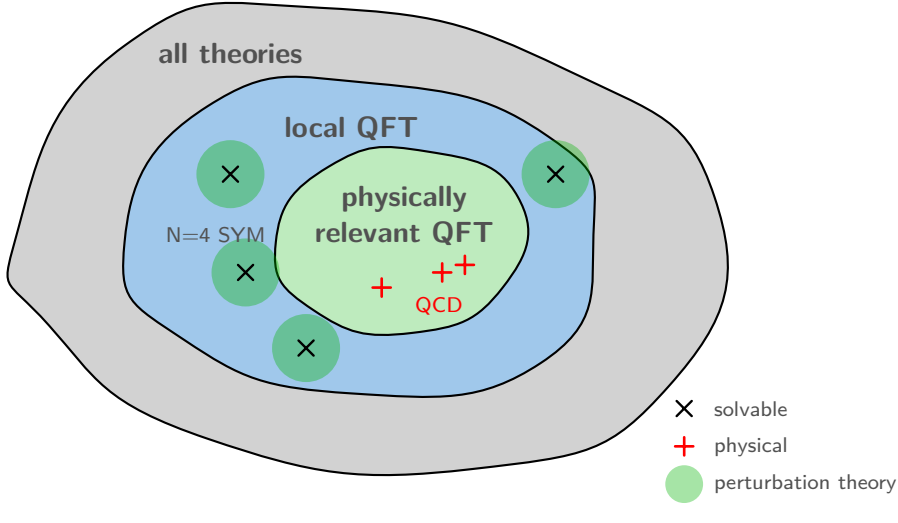


$\mathcal{N} = 4$ *SYM* - Chrysler building

The shape of theory space



The shape of theory space



The toy model

The best starting point: ϕ_2^4

Renormalized ϕ_2^4 theory

$$H = \int \mathrm{d}x \, \frac{:\pi^2:}{2} + \frac{:(\nabla\phi)^2:}{2} + \frac{m^2}{2} :\phi^2: + g:\phi^4:$$

with $[\phi(x), \pi(y)] = i\delta(x-y)$

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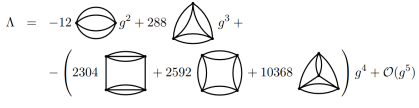
with $[\phi(x), \pi(y)] = i\delta(x-y)$

1. Rigorously defined relativistic QFT without cutoff (Wightman QFT)
2. Vacuum energy density ε_0 finite for all g
3. Difficult to solve unless $g \ll m^2$ – not integrable
4. Phase transition around $f_c = \frac{g}{4m^2} = 11$ i.e. $g \simeq 2.78$ in mass units

Historical approaches can be used

Perturbation theory

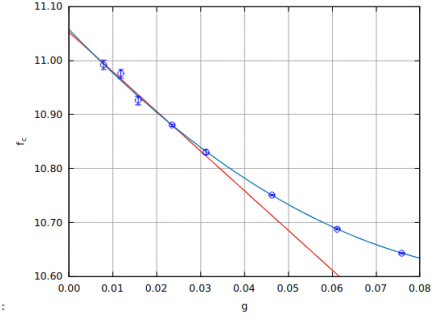
+ resummation



state of the art is $O(g^8)$

arXiv:1805.05882
Serone, Spada, Villadoro

Lattice Monte-Carlo



arXiv:1807.03381
Bronzin, De Palma, Guagnelli

The variational method

The variational method

Variational method for ground state search

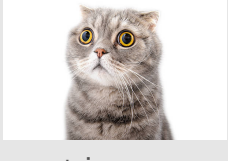
1. Guess a manifold $\mathcal{M} \subset \mathscr{H}$ with few parameters ν *i.e.* $\dim\mathcal{M} \ll \dim\mathscr{H}$
2. Tune ν to minimize energy $\nu = \operatorname{argmin}_{\nu\in\mathcal{M}} \frac{\langle\nu|H|\nu\rangle}{\langle\nu|\nu\rangle}$ and get $|\text{ground state}\rangle \simeq |\nu\rangle$

The variational method

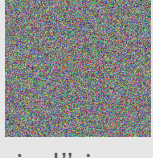
Variational method for ground state search

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Reason for compression (classical)



cat image



"typical" image

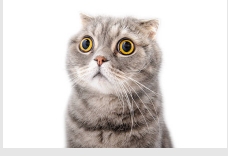
atypical $\Longrightarrow$ compressible

The variational method

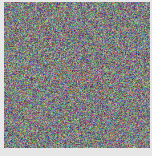
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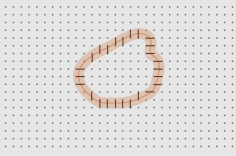
cat image



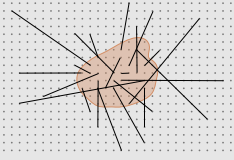
"typical" image

atypical $\Longrightarrow$ compressible

Reason for compression (quantum)



low energy state



random state

area law = atypical $\Longrightarrow$ compressible

Feynman's criticism

Difficulties in Applying the Variational Principle to Quantum Field Theories[1]

so I tried to do something along these lines with quantum chromodynamics. So I'm talking on the subject of the application of the variational principle to field theoretic problems, but in particular to quantum chromodynamics.

I'm going to give away what I want to say, which is that I didn't get anywhere! I got very discouraged and I think I can see why the variational principle is not very useful. So I want to take, for the sake of argument, a very strong view – which is stronger than I really believe – and argue that it is no damn good at all!

Feynman's criticism

Difficulties in Applying the Variational Principle to Quantum Field Theories[1]

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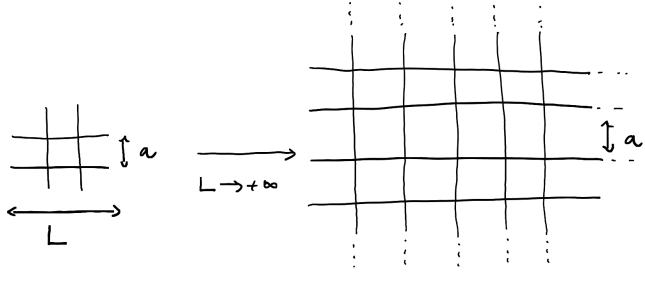
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Feynman's requirement in a nutshell

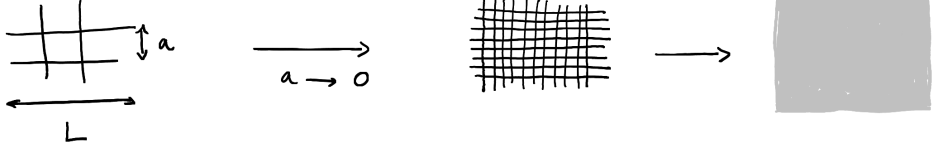
1. **Extensive parameterization** ← IR problem
   Number of parameters $\propto L^\alpha$ at most for system size L (not $\propto \mathrm{e}^L$)
2. **Computable expectation values**
   ψ known $\implies \langle \mathcal{O}(x)\mathcal{O}(y)\rangle_\psi$ computable
3. **Not oversensitive to short distances** ← UV problem
   no runaway minimization where higher and higher momenta get fitted

UV and IR problems, from the lattice perspective

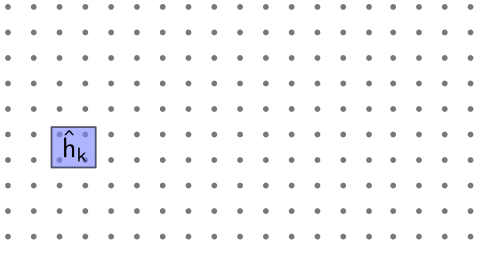
1. **IR problem**: taking thermodynamic limit, keeping H_{latt} fixed



2. **UV problem**: taking small lattice spacing, while scaling H_{latt}



Quantum many-body problem on the lattice



Typical condensed matter problem = discretized QFT

$$|\psi\rangle = \sum c_{i_1,i_2,\cdots,i_n} |i_1, i_2, \cdots i_N\rangle$$

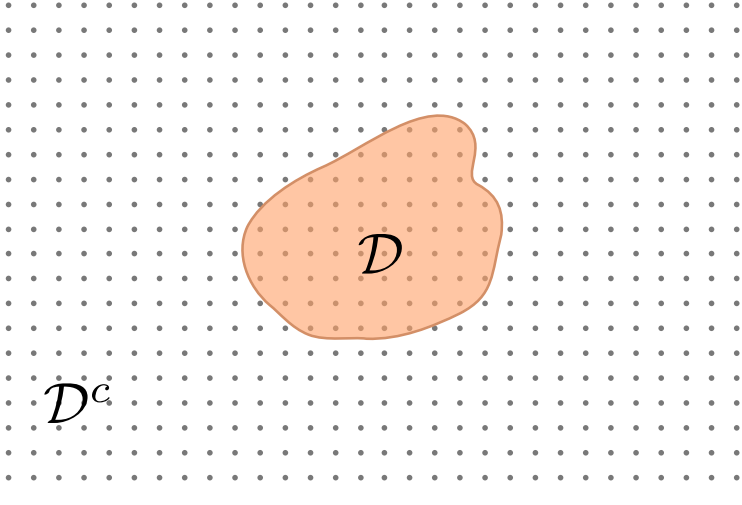
Problem:

Finding the low energy states of

$$\hat{H} = \sum_{k=1}^{N} \hat{h}_k$$

is **hard** because $\dim \mathscr{H} \propto 2^N$ for spins

Interesting states are weakly entangled



Low energy state

$|\psi\rangle = |0\rangle$ or $|1\rangle$ ...

Reduced density matrix

$\rho = \mathrm{tr}_{\mathcal{D}^c}\left[|\psi\rangle\langle\psi|\right]$

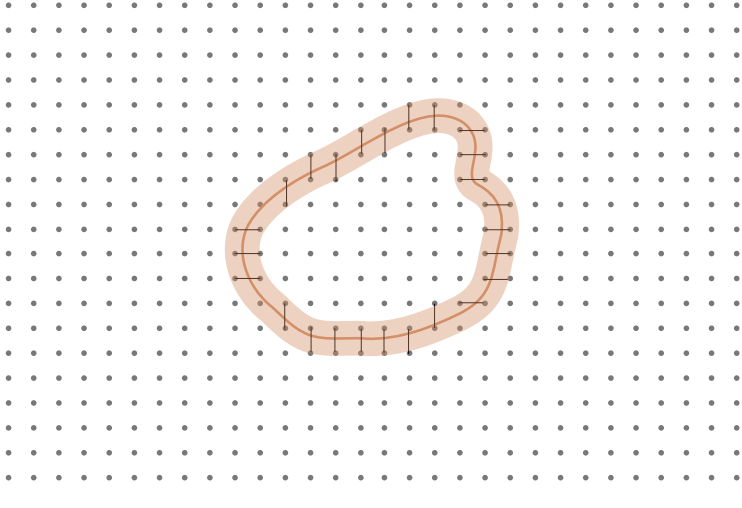
Entanglement entropy

$S = -\mathrm{tr}\left[\rho \log \rho\right]$

Area law

$S \propto |\partial\mathcal{D}|$

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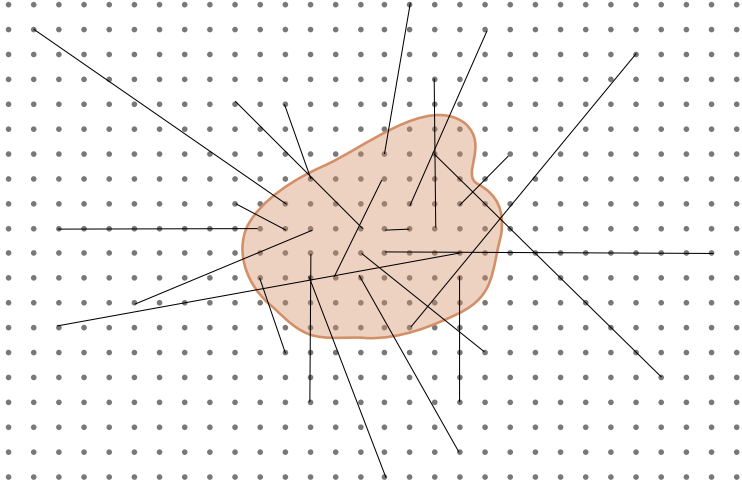
Entanglement entropy

$S = -\mathrm{tr}\left[\rho \log \rho\right]$

Area law

$S \propto |\partial\mathcal{D}|$

Typical states are strongly entangled



Random state

$|\psi\rangle = U_{\text{Haar}}|\text{trivial}\rangle$

Reduced density matrix

$\rho = \text{tr}_{\mathcal{D}^c}\left[|\psi\rangle\langle\psi|\right]$

Entanglement entropy

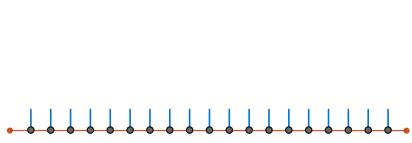
$S = -\text{tr}\left[\rho \log \rho\right]$

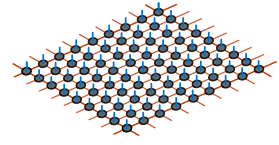
Volume law

$S \propto |\mathcal{D}|$

Tensor network states target the area law region

.zip or **.jpg** for complex quantum states that appear in Nature





1. **Exponential reduction:** $2^N \longrightarrow N \times D^{2d}$ parameters
   [N number of spins, D amount of entanglement, d space dimension $(1, 2, 3)$]

2. **Efficient compression:** compression error $\leqslant e^{-D}$ or $1/\text{superpoly(D)}$
   [For a large number of *a priori* non-trivial problems]

[History: 1992 for $d = 1$, 2004 for $d \geqslant 2$, 2016 for 2d-Hubbard at $T = 0$]

Feynman's checklist for tensor network states

1. **Extensive parameterization $\rightarrow$ OK**
2. **Computable expectation values $\rightarrow$ OK**
3. **Not oversensitive to short distances $\rightarrow$??????**

Just take the continuum limit to solve **3**?

Similarity between relativistic and critical models

- A critical model is scale invariant in the IR

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \underset{|x-y|\to+\infty}{\sim} \frac{1}{|x-y|^{2\Delta_{\mathcal{O}}}}$$

- A relativistic QFT is scale invariant in the UV

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \underset{|x-y|\to 0}{\sim} \frac{1}{|x-y|^{2\Delta_{\mathcal{O}}}}$$

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Consequence on entanglement

With a UV cutoff $\Lambda = 1/a$ in $1+1$ dimensions:

$$S \propto \log(\Lambda) \implies D \propto \Lambda^{\kappa}$$

$\implies$ infinite amount of information in high frequency modes

UV “criticality” is usually milder than IR criticality

UV CFT tend to be kind

For QFT that are either

1. **super renormalizable** or
2. **asymptotically free**

the critical behavior at short distance is **free**

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UV CFT tend to be kind

For QFT that are either

1. **super renormalizable** or
2. **asymptotically free**

the critical behavior at short distance is **free**

E.g. for ϕ_2^4 at short distances

$$H \longrightarrow H_0 = \int \mathrm{d}x \, \frac{:\pi^2:_m}{2} + \frac{:(\nabla\phi)^2:_m}{2} + \frac{m^2}{2} :\phi^2:_m = \frac{1}{2\pi} \int \mathrm{d}k \, \omega_k a_k^\dagger a_k$$

which is exactly solvable – disentangled modes.

Relativistic continuous matrix product states

- Variational method in Quantum field theory without cutoff
- Relativistic continuous matrix product states for quantum fields without cutoff

Relativistic continuous matrix product states

"continuum limit of an MPS written in the Fourier transformed modes diagonalizing/disentangling the free part of the Hamiltonian"

Definition

RCMPSs are a manifold of states parameterized by 2 $(D \times D)$ matrices Q, R

$$|Q,R\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q\otimes\mathbb{1} + R\otimes a^\dagger(x)\right]\right\}|0\rangle$$

with

- $|0\rangle$ is the Fock vacuum of the free part H_0 of H
- trace taken over $\mathbb{C}^D$
- $\mathcal{P}$ path-ordered exponential
- $a(x) = \frac{1}{2\pi}\int \mathrm{d}k\, e^{ikx} a_k$

Basic properties of RCMPS

$$|Q,R\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q\otimes\mathbb{1} + R\otimes a^\dagger(x)\right]\right\}|0\rangle$$

Feynman's checklist:

1. **Extensive** because of $\mathcal{P}\exp\int$
2. Observables **computable** at cost D^3 (non trivial!)
   requires $[a(x), a^\dagger(y)] = \delta(x-y)$
3. **No UV problems**
   $|0,0\rangle = |0\rangle$ is the ground state of H_0 hence exact CFT UV fixed point
   $\langle Q,R| : P(\phi) : |Q,R\rangle$ is finite for all Q, R (not trivial!)

The variational algorithm

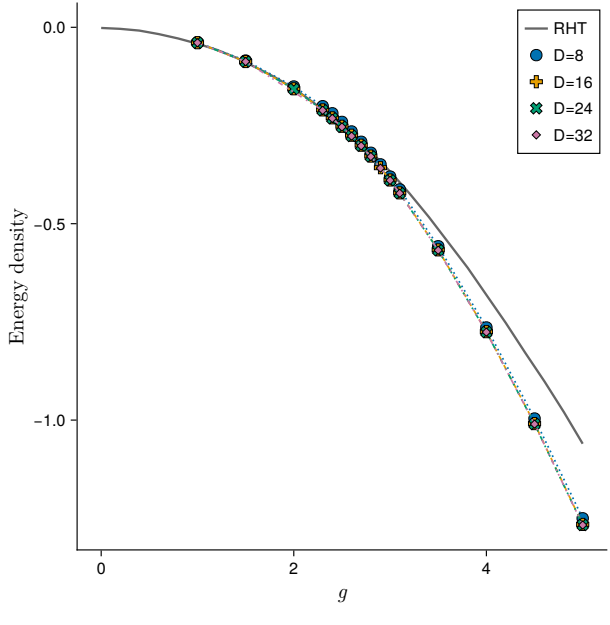
Optimization

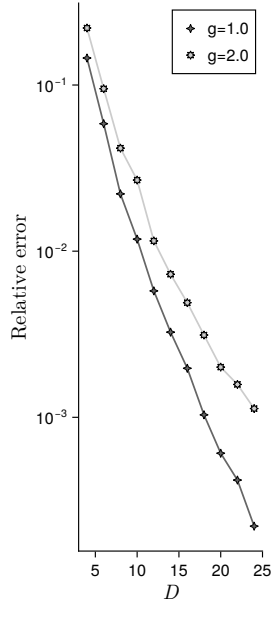
Compute $e_0 = \langle Q, R|h|Q, R\rangle$ and $\nabla_{Q,R} e_0$
Minimize e_0 with Riemannian GD $=$ projected imaginary time evolution

Cost of e_0 and ∇e_0 :

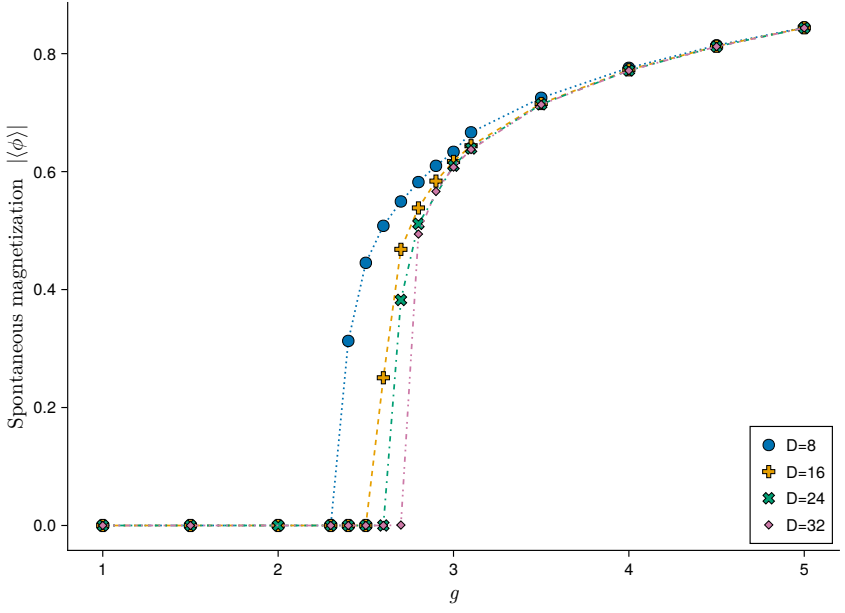
1. $e_0 = \langle h \rangle_{QR}$ computable at cost $O(D^3)$
2. ∇e_0 computable at cost $O(D^3)$ via backpropagation

First results: energy density

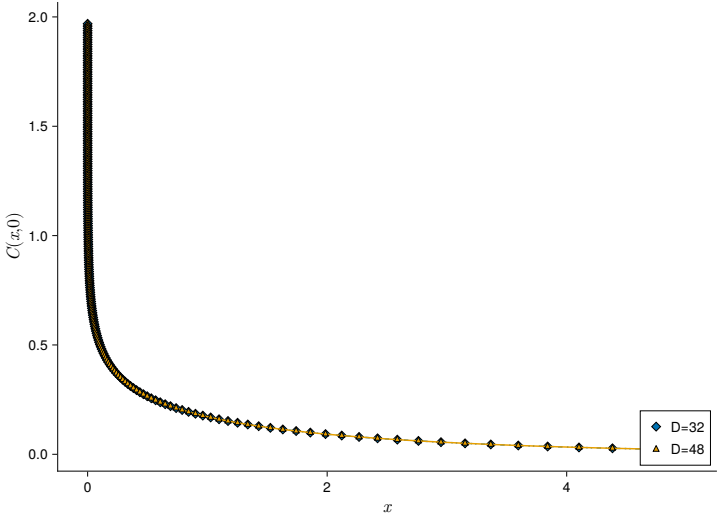




First results: spontaneous magnetization



Precise results: 2-point function



For $g = 2$

Beyond local observables

- *Extracting quantum field theory dynamics from an approximate ground state* with **Sophie Mutzel**
- *A relativistic continuous matrix product state study of field theories with defects* with **Karan Tiwana** and **Edo Lauria**

How much can we get from the ground state?

We have $|Q, R\rangle = |\text{vac}\rangle$ to exquisite precision, now what?

Available data

All **equal-time** correlation functions:

$$C(x_1, x_2, \cdots, x_n) = \langle Q, R | \, \phi(x_1)\, \phi(x_2) \cdots \phi(x_n) \, | Q, R \rangle$$

still at cost $O(D^3)$

2-point functions

Relativistic (Euclidean) invariance is useful for once:

$$\langle \phi(x_1,\tau_1)\phi(x_2,\tau_2)\rangle = \langle \phi\big(\sqrt{(x_1-x_2)^2+(\tau_1-\tau_2)^2},0\big)\phi(0,0)\rangle$$

Källén-Lehman spectral representation

$$\langle \phi(x)\phi(0)\rangle = \int_{\mathbb{R}^+} \mathrm{d}s\, \rho(s)\, K_0(s|x|)$$

where:

1. $\rho(s) \geqslant 0$ for all s
2. $K_0(s|x|)$ is the *free* 2-point function at mass $m^2 = s$

For a $\mathbb{Z}^2$-symmetric scalar field: $\rho(s) = Z_\phi \delta(s-M^2) + \theta(s-9M^2) \times \rho_{\text{reg}}(s)$

Bootstrap the mass

Assume $\rho(s) = Z_\phi \delta(s - M^2) + \theta(s - 9M^2) \times \rho_{\text{reg}}(s)$ with M fixed. Is it compatible with

$$\langle Q, R|\phi(x)\phi(0)|Q, R\rangle \simeq \int_{\mathbb{R}^+} \mathrm{d}s\, \rho(s)\, K_0(s|x|) \quad \text{for} \quad x \in \{x_1, x_2, \cdots, x_N\} \quad ?$$

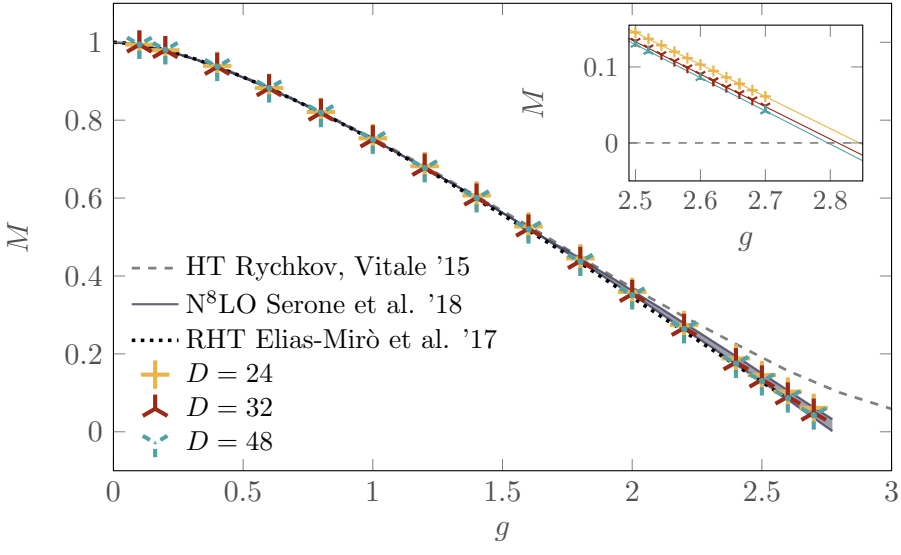
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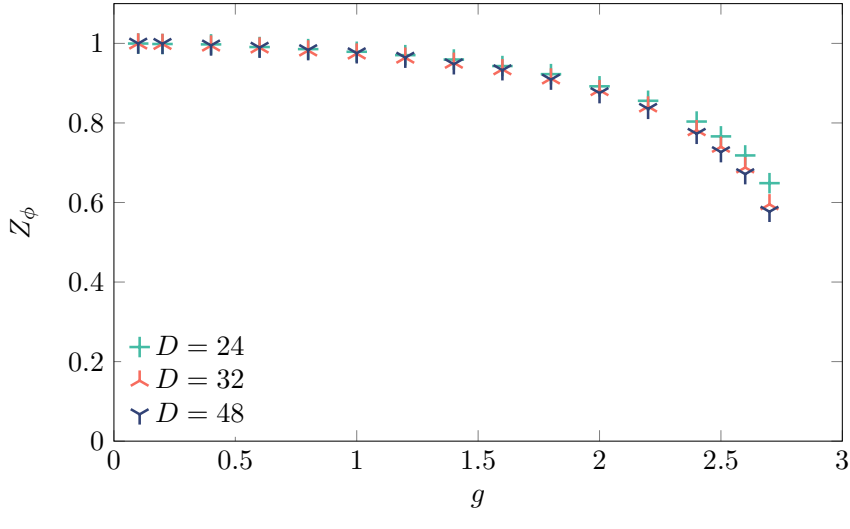
1. This questions is a *linear program* $\rightarrow$ efficiently answerable
2. The answer is "NO" for all M except $M \in [M_{\text{est}} - \varepsilon, M_{\text{est}} + \varepsilon]$
3. Means $M \simeq M_{\text{est}}$

Mass for ϕ_2^4

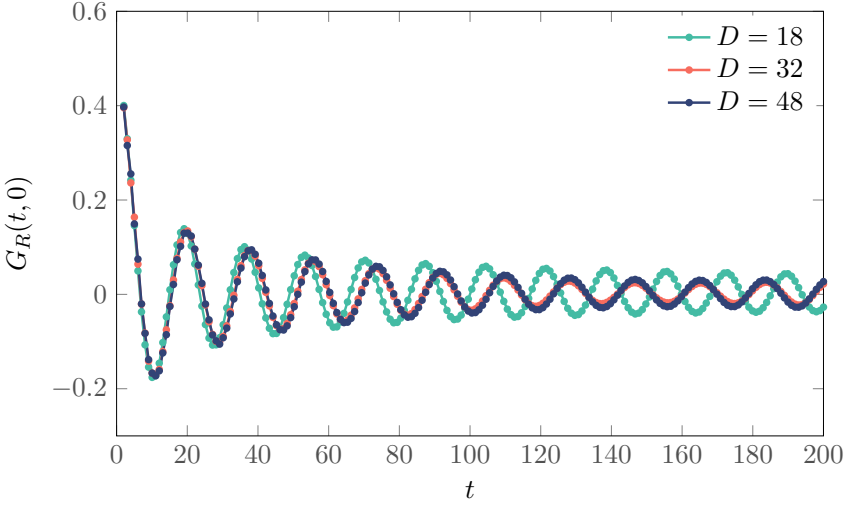


This implies $g_c \simeq 2.796$, compatible with the best lattice extrapolations

Wave-function renormalization



Real-time propagator



Beyond toy models

- *A study of the quantum Sinh-Gordon model with relativistic continuous matrix product states*
- *Multi-Field Relativistic Continuous Matrix Product States*
  with **Karan Tiwana**

After 1 scalar field, 2 scalar fields

Beyond the pile of dirt ϕ_2^4:

$$H = H_0(\phi_1) + H_0(\phi_2) + V(\phi_1, \phi_2)$$

with quartic potential

$$V(\phi_1, \phi_2) = \int_{\mathbb{R}} \mathrm{d}x \; g \left(:\phi_1^4: + :\phi_2^4: \right) + 2\lambda : \phi_1^2 \phi_2^2 :$$

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Much richer physics

- D_4 symmetry $\implies$ complicated phase diagram similar to Ashkin-Teller
- Enhanced symmetry $D_4 \to O(2)$ on the parameter line $g = \lambda$
- BKT transition

Multi-field RCMPS

Ansatz for two fields:

$$|Q, R_1, R_2\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q \otimes \mathbb{1} + R_1 \otimes a_1^\dagger(x) + R_2 \otimes a_2^\dagger(x)\right]\right\}|0\rangle$$

Multi-field RCMPS

Ansatz for two fields:

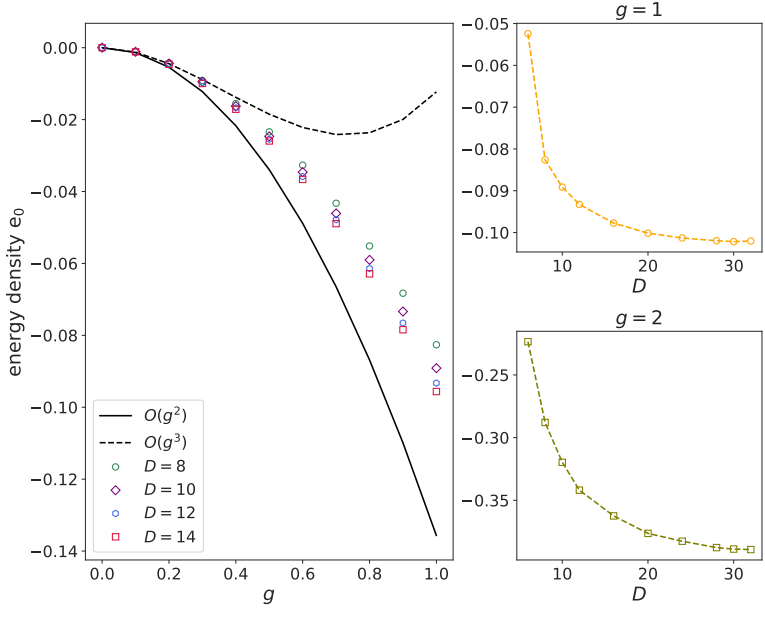
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But subtle **regularity constraint**

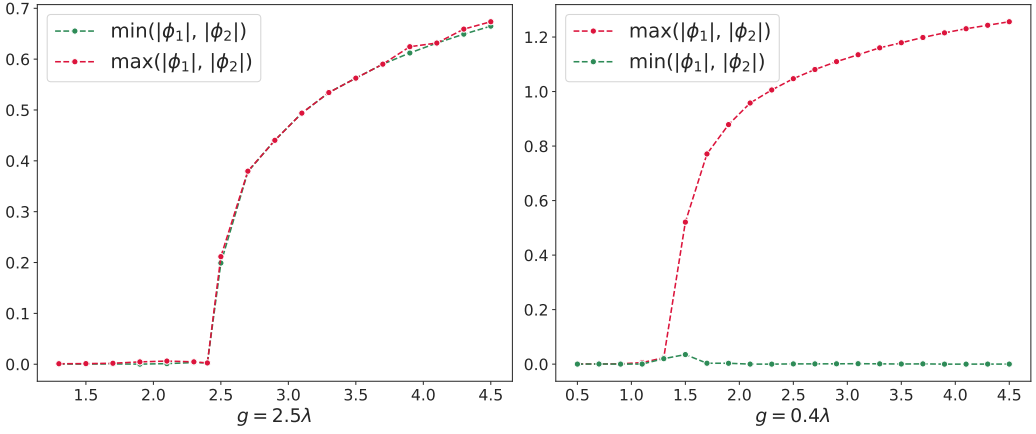
$$\langle Q, R_1, R_2|\, h\, |Q, R_1, R_2\rangle < +\infty \quad \Longrightarrow \quad [R_1, R_2] = 0$$

- Need to optimize over Q, R_1, R_2 under the constraint $[R_1, R_2] = 0$
- Requires new Riemannian optimization tricks
- Still $O(D^3)$

Results: energy density



Results: two phases (among many)



Results at $D = 16$

Some problems ahead

Extensions

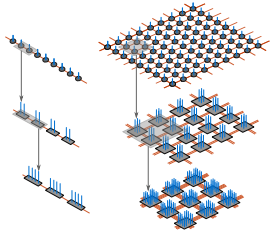
Natural follow up

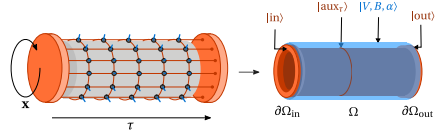
- Estimate more complex observables (like S-matrix elements)
- Full phase diagram for deformations of $O(2)$, $O(3)$, etc.
- Study models with Fermions/Gauge: Schwinger model [subtleties]
- Finite entanglement scaling
- Push D with better optimizer for "exact" results

Challenges

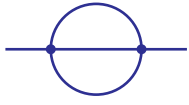
Main challenge: beyond $1+1$ dimensions

- Difficult on the tensor network side





- Difficult on the QFT side $\langle h \rangle = +\infty$



$$= \frac{\lambda^2}{6} \int^{\Lambda} \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{k^2+m^2} \frac{1}{q^2+m^2} \frac{1}{(p-k-q)^2+m^2}$$

$$= \frac{\lambda^2}{384\pi^2} \log\left(\frac{\Lambda}{m}\right) + \text{finite terms}\,.$$

Should we keep on?

Summary

New method to solve some QFT in $1+1$ dimensions without cutoff, UV or IR

$$|\mathrm{vac}\rangle \simeq |Q,R\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q\otimes\mathbb{1} + R\otimes a^\dagger(x)\right]\right\}|0\rangle$$

- Almost numerically exact for local observables in massive phase (cost D^3, error plausibly $1/\mathrm{superpoly}(D)$)
- Can be extended to multiple interacting scalar fields
- Rigidity of QFT allows to get M, Z_ϕ, real-time propagator from $|Q,R\rangle$