Spikes in quantum trajectories
and how to blow them up

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Introduction

When a quantum system is continuously monitored in a basis which is not preserved by its proper evolution, an interesting competition arises between free evolution and measurement. When the continuous measurement part dominates, the system undergoes quantum jumps between measurement pointer states. This phenomenon is ubiquitous, but there is an additional less known subtlety: sharp fluctuations around the jumps, dubbed spikes, persist even when the measurement process fully dominates the dynamics. Essentially, this means that a system subjected to a strong continuous measurement does not behave exactly as expected, i.e. as if it were subjected to repeated standard Von-Neumann measurements. We propose to illustrate this phenomenon on an example.

Main equation

A 1-variable quantum trajectory equation which captures the quintessential subtlety of the competition between measurement and evolution is that of a qubit coupled to a thermal bath and subjected to the continuous measurement of its energy. In such a problem, the only relevant quantity (see box below) is the probability Q_t to be in the ground state at time t which obeys:

$$dQ_t = \underbrace{\lambda(p_{eq} - Q_t)\,dt}_{\text{effect of the bath}} + \underbrace{\sqrt{\gamma}\,Q_t(1-Q_t)\,dW_t}_{\text{effect of the measurements}} \quad (1)$$

where λ is the coupling with the bath, p_{eq} the equilibrium probability, γ the measurement rate and W_t is a Wiener process.

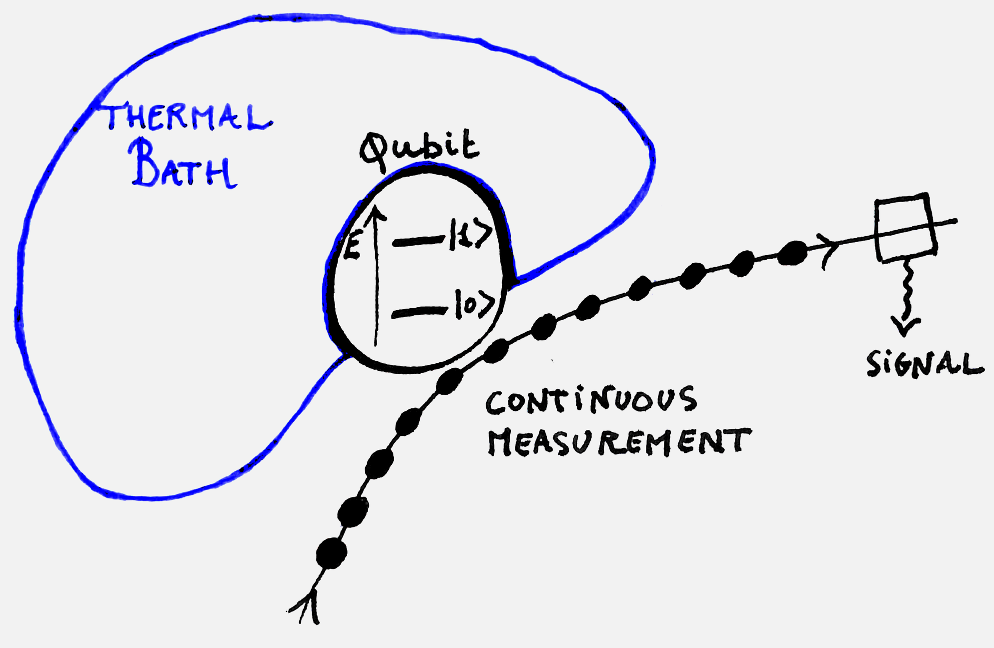
Intuitively:

- $\lambda(p_{eq} - Q_t)\,dt$ drags the system towards the equilibrium Boltzmann probability p_{eq}.
- $\sqrt{\gamma}\,Q_t(1-Q_t)\,dW_t$ drags the system towards $Q = 0$ or $Q = 1$, i.e. perfect certainty.

We are interested in the behavior of this equation when the continuous measurement becomes strong, i.e. when $\gamma \to +\infty$ (with λ fixed).

Eq. (1) from continuous measurement theory

Model



$$d\rho_t = d\rho_t^{bath} + d\rho_t^{meas} \quad (2)$$

Effect of the bath

$$d\rho_t^{bath} = \lambda p_{eq}\left(\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}\right) + \lambda(1-p_{eq})\left(\sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho\}\right)$$

Effect of the continuous measurement

$$d\rho_t^{meas} = \gamma\mathcal{D}[\sigma_z/2](\rho_t)\,dt + \sqrt{\gamma}\mathcal{H}[\sigma_z/2](\rho_t)\,dW_t$$

with:

$$\mathcal{D}[M](\rho) = M\rho M^\dagger - \frac{1}{2}\{M^\dagger M, \rho_t\}$$

$$\mathcal{H}[M](\rho) = M\rho + \rho M^\dagger - \rho\,\mathrm{Tr}\left[M\rho + \rho M^\dagger\right]$$

Expanding (2) shows that $Q_t = \langle 0|\rho|0\rangle$ verifies exactly eq. (1). Additionally the non-diagonal coefficients are suppressed exponentially fast and can thus be neglected.

Jumps and Spikes

Equation (1) gives rise to two interesting phenomena in the large γ limit: jumps [2] and spikes [1].

Jumps

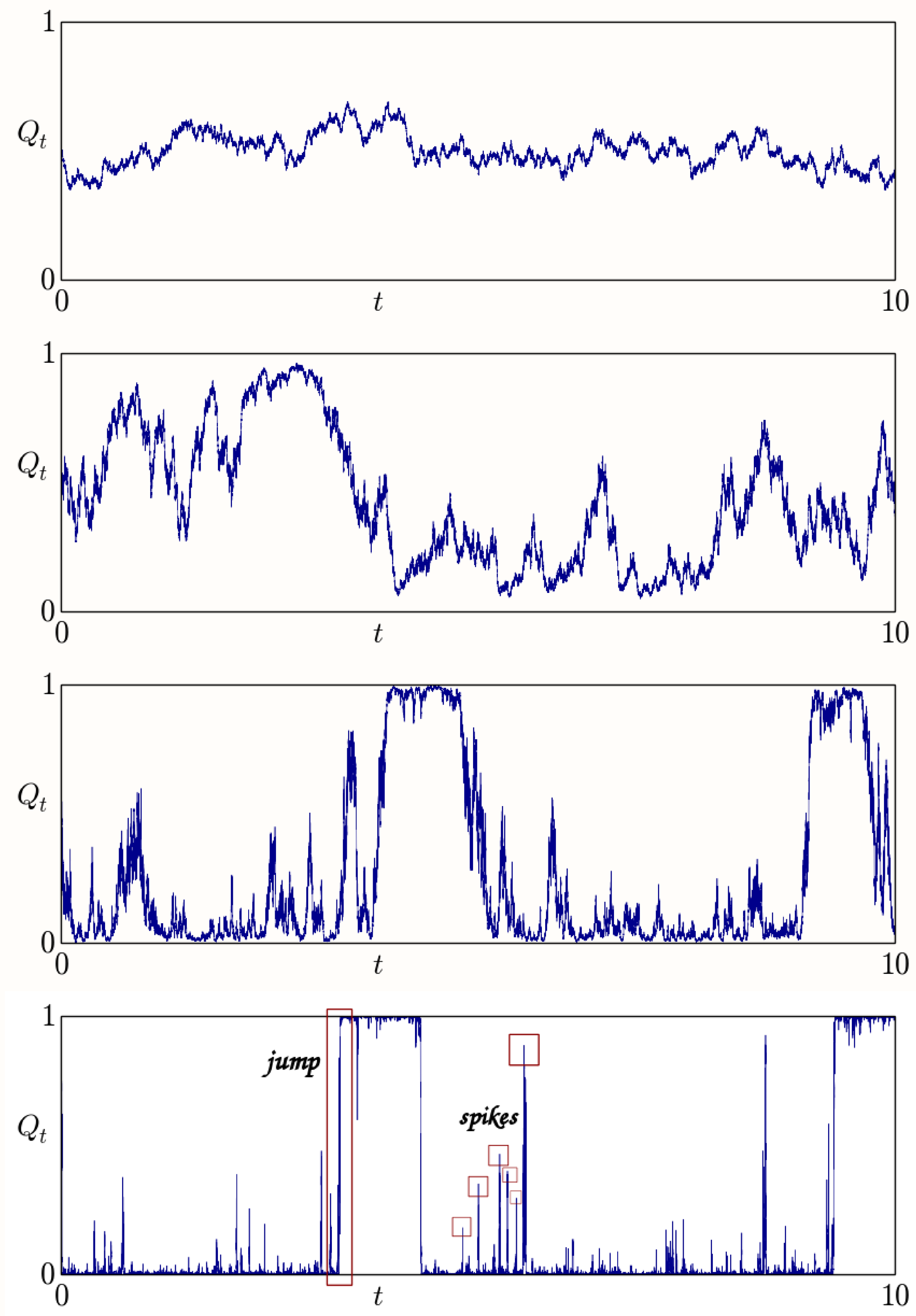


Figure: Quantum trajectories of Q_t for increasing values of γ ($\gamma = \{0.1, 1, 10, 250 \simeq +\infty\}$). In the last figure, one can see distinctively the presence of **jumps**, i.e. fast excursions $0 \to 1$ or $1 \to 0$ and **spikes**, i.e. fast excursions $0 \to 0$ and $1 \to 1$.

Spikes

The spikes, which can be seen in the previous figure, become sharper and sharper when $\gamma \to +\infty$ but never disappear and can be quantified in the limit. Consider for simplicity the spikes starting from 0.

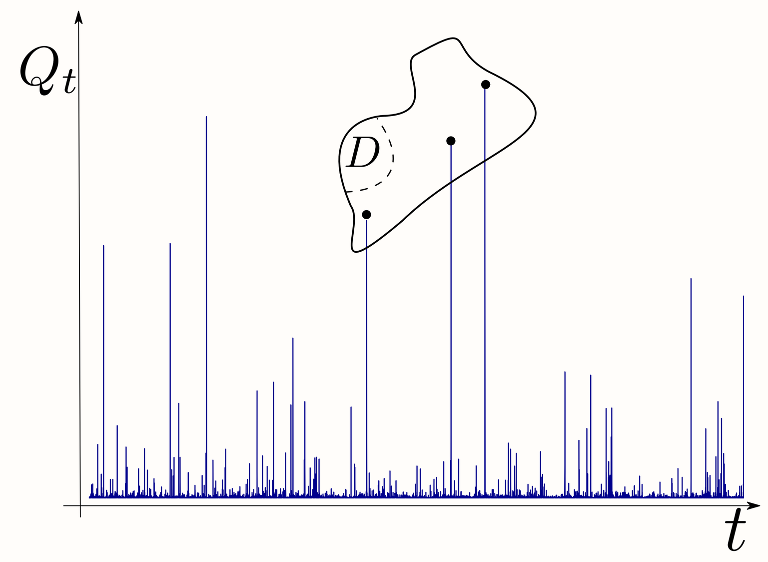


Figure: Quantifying spikes

The number of spikes in a domain D (where there is no jump) can be shown to be [1] a Poisson process of intensity $I = \int_D d\nu$ with:

$$d\nu = \frac{\lambda p}{Q^2}\,dt\,dQ$$

Unfolding time

Spikes and jumps spoil the well-definiteness of the limit $\gamma \to +\infty$. The solution is to redefine the time to blow them up and resolve their inner structure. A global rescaling would not do the trick because we would need an infinite amount of time to see Q vary: we have to act locally.

Define the effective time

$$\tau(t) := \int_0^t (dQ_u)^2 = \gamma\int_0^t Q_u^2(1-Q_u)^2 du \quad (3)$$

With this new time parametrisation, things are well defined when $\gamma \to +\infty$ and the limiting process takes a very simple form.

Proposition

When $\gamma \to +\infty$:

(i) Q_τ is a Brownian motion reflected at 0 and 1.

(ii) The linear time t can the be expressed as a function of the effective time τ:

$$t(\tau) = \frac{L_\tau}{\lambda p} + \frac{U_\tau}{\lambda(1-p)} \quad (4)$$

where L_τ and U_τ are the *local times* spent by Q_τ respectively in 0 and 1.

Local time For a Brownian like process X_t, the local time L_t at 0 is defined formally by $L_t := \int_0^t dt'\delta(X_{t'})$. Intuitively, the local time in 0 represents the rescaled time the process spends in 0.

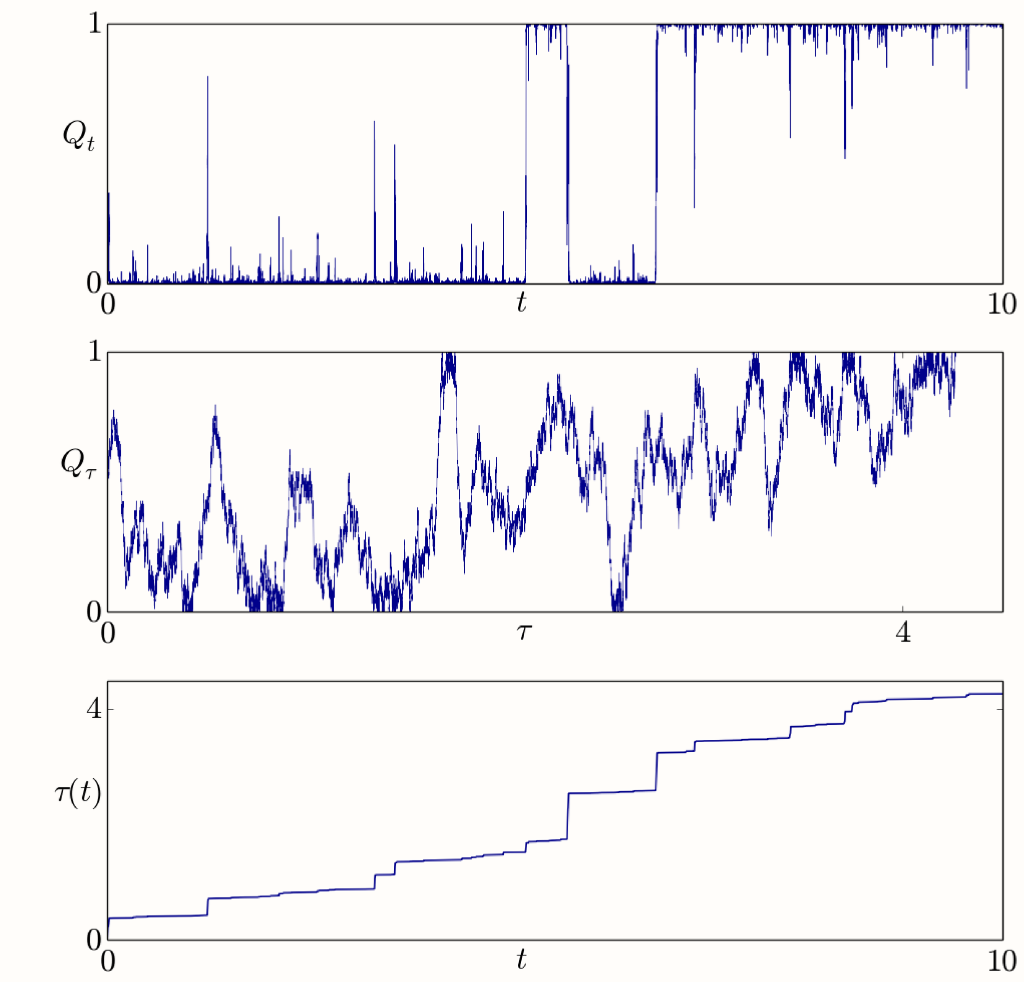


Figure: Trajectories of Q in real time and effective time. In real time, the trajectory is spiky, with infinitely sharp fluctuations. In effective time, the trajectory is a reflected Brownian motion where the inner structure of the spikes is unfolded. The last plot shows the effective time as a function of the real time which increases like a Devil's staircase, only where there are spikes.

Example of the linear entropy

$$S^L = 1 - \mathrm{Tr}[\rho^2] = 2\,Q(1-Q)$$

In real time t, it verifies the SDE:

$$dS_t^L = 2\lambda(1-2Q_t)(p-Q_t)\,dt + Q_t(1-Q_t)\left[2\sqrt{\gamma}(1-2Q_t)dW_t - 2\gamma Q_t(1-Q_t)dt\right]$$

which has no well defined limit when $\gamma \to +\infty$.

However, introducing the effective time τ and using the proposition one gets:

$$dS_\tau^L = 2(1-2Q_\tau)\,dB_\tau - 2\,d\tau + 2(dL_\tau + dU_\tau) \quad (5)$$

which shows a non trivial competition between measurement ($-2\,d\tau + 2(dL_\tau + dU_\tau)$) and the bath ($2(1-2Q_\tau)\,dB_\tau$). This level of details would have been lost taking naively the limit in real-time : the linear entropy is an *anomalous* observable.

Conclusion

Strongly monitored quantum systems display a very unintuitive behavior with spikes in addition to the expected jumps. Nevertheless, using a time redefinition, it is possible to capture this singular phenomenon and to do exact computations in the limit.

References

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