Ballistically induced diffusion in open quantum random walks

Antoine Tilloy, under the supervision of Denis Bernard and Michel Bauer

ENS Paris, Laboratoire de Physique Théorique

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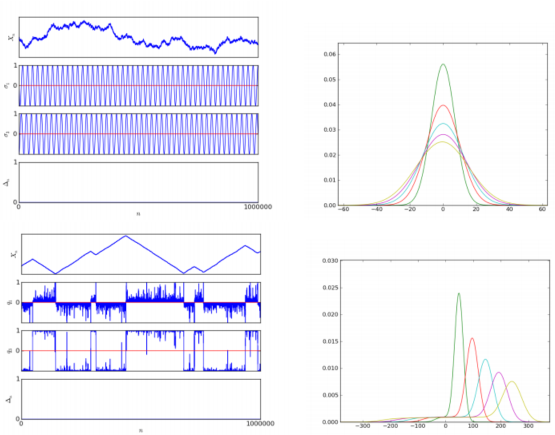
The main problem of this talk is to understand the regime switch in the simplest OQRW model.



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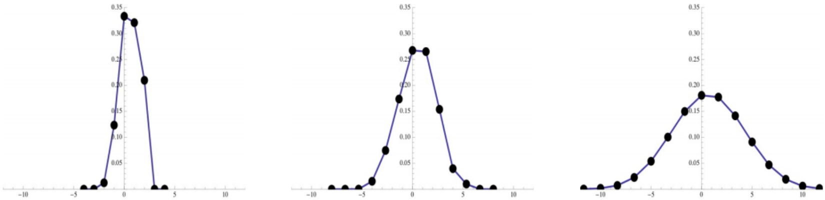
Based *Open quantum random walks* by S. Attal, F. Petruccione C. Sabot and I. Sinayskiy



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- Completely positive map defined with the help of B_+ and B_-:

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$$\downarrow$$
$$B_+ \rho_0 B_+^\dagger \otimes |i+1\rangle\langle i+1| + B_- \rho_0 B_-^\dagger \otimes |i-1\rangle\langle i-1|$$

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- Can be dilated on $\mathbb{C}^2 \otimes \mathbb{C}^{\mathbb{Z}} \otimes \left(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots\right)$ to get a physical picture.

After two iterations one gets:

$$B_+^2 \rho_0 B_+^{\dagger\,2} \otimes |i+2\rangle\langle i+2| + \left(B_+ B_- \rho_0 B_-^\dagger B_+^\dagger + B_- B_+ \rho_0 B_+^\dagger B_-^\dagger\right) \otimes |i\rangle\langle i| + B_- \rho_0 B_-^\dagger \otimes |i-2\rangle\langle i-2|$$

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So the probabilities to be in $i+2$, i and $i-2$ at time 2 are respectively:

$$Tr\ B_+^2 \rho_0 B_+^{\dagger\,2} \quad (1)$$

$$Tr\ \left(B_+ B_- \rho_0 B_-^\dagger B_+^\dagger + B_- B_+ \rho_0 B_+^\dagger B_-^\dagger\right) \quad (2)$$

$$Tr\ B_-^2 \rho_0 B_-^{\dagger\,2} \quad (3)$$

which cannot be expressed in terms of the previous probabilities only

Tautology

What happens if one measures the trajectory but does not read the result before the end of the experiment ?
Let us imagine that at time t=n, the density matrix of the system is:

$$\rho^n = \sum_i \rho_i |i\rangle\langle i|$$

If the position is measured but not read, the density matrix is now:

$$\frac{1}{Tr(\rho_i)}\rho_i|i\rangle\langle i| \;\; \textit{with probability} \;\; Tr(\rho_i)$$

i.e. by definition of the density matrix, the system is described by the density matrix:

$$\rho^{n+} = \sum_i \frac{Tr(\rho_i)}{Tr(\rho_i)}\rho_i|i\rangle\langle i| = \sum_i \rho_i|i\rangle\langle i| = \rho^n$$

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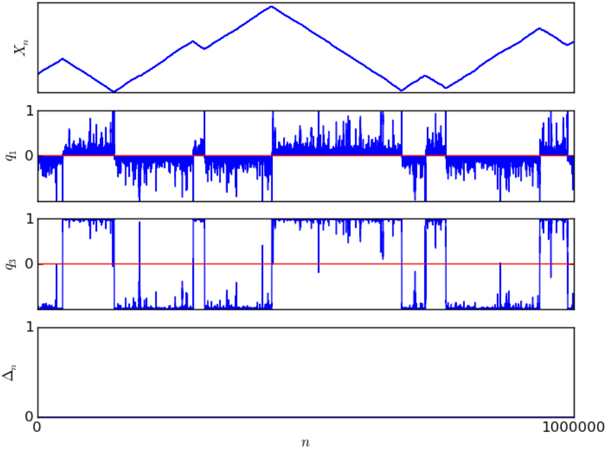
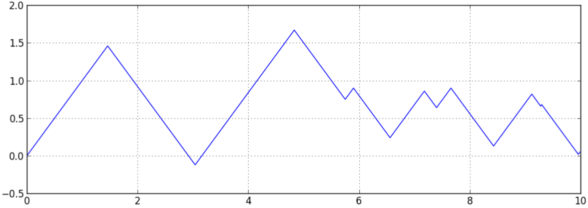
$\longrightarrow$ Trajectories (or realizations) of OQRW have a sense and can be studied in contrast with what happens with UQW.


X_n
1
q_1
0
1
q_3
0
1
Δ_n
0
0
1000000
n

Short Break: study of a toy model

- Reverse-engineer the weird behavior of our OQRW in an even simpler case
- Understand the modification of the diffusion constant

What would be the Fokker-Planck equation of such an idealized process ?



Slope ± 1 with a rate of change λ

This process if of course non Markovian: we need to define $p_+(x,t)$ and $p_-(x,t)$ the probabilities to be in x at time t with respectively positive and negative slope.

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$$\begin{aligned} \partial_t p_+ + \partial_x p_+ - \lambda(p_+ - p_-) = 0 \\ \partial_t p_- - \partial_x p_- - \lambda(p_- - p_+) = 0 \end{aligned} \tag{4}$$

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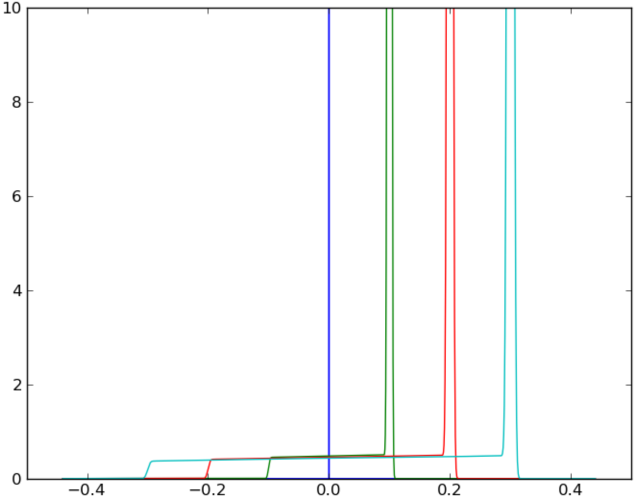
i.e.

$$\partial_t P = -\sigma_z \partial_x P + \lambda(\mathbb{I} - \sigma_x) P \tag{5}$$

Results:



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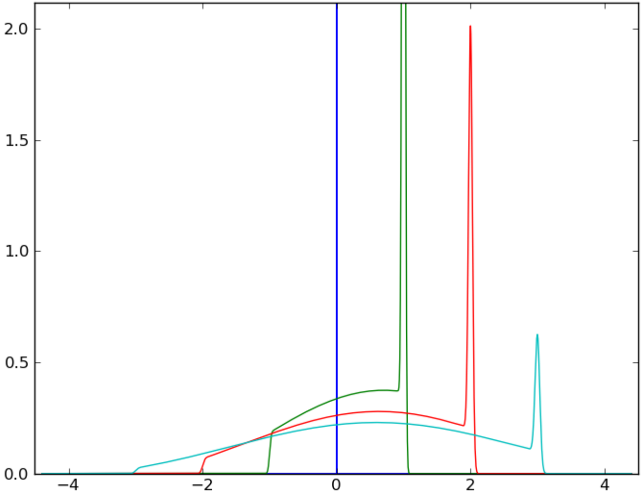

- No need for a Laplacian term to converge to a Gaussian at large time
- The matrix σ_x couples the two ballistic evolution in the Fokker-Planck picture and allows the regime switch.

We want to find the most general scaling limit to be sure not to forget anything at the continuous limit. We try to find B_+ and B_- in the following form:

$$\begin{aligned} B_+ &= \frac{1}{\sqrt{2}}\left(\mathbb{I} + \sqrt{\epsilon}N_+ + \epsilon M_+ + o(\epsilon)\right) \\ B_- &= \frac{1}{\sqrt{2}}\left(\mathbb{I} + \sqrt{\epsilon}N_- + \epsilon M_- + o(\epsilon)\right) \end{aligned} \tag{6}$$

And we write the full density matrix:

$$\rho = \int dx\, \rho(x,t)\, |x\rangle\langle x| \tag{7}$$

Then we keep on the dirty way writing:

$$\rho(x, t + dt) = \ B_{-}\rho(x + dx, t)B_{-}^{\dagger} + \ B_{+}\rho(x - dx, t)B_{+}^{\dagger}$$

And the previous expression for the B's gives:

$$\begin{aligned}\frac{\partial}{\partial t}\rho(x,t) =& \frac{1}{dt}\left(-\rho(x,t) + \frac{\rho(x+dx,t)+\rho(x-dx,t)}{2}\right)\\ &+ \frac{\sqrt{\epsilon}}{2dt}\left(N_{+}\rho(x-dx,t) + N_{-}\rho(x+dx) + \rho(x-dx,t)N_{+}^{\dagger} + \rho(x+dx)N_{-}^{\dagger}\right)\\ &+ \frac{\epsilon}{2dt}\,(...)\end{aligned}$$

Constraint:

- keep the maximum number of non diverging terms
- verify $B_+^\dagger B_+ + B_-^\dagger B_- = \mathbb{I}$ up to order ϵ

This gives:

Scaling

$$B_+ = \frac{1}{\sqrt{2}} \left(\mathbb{I} + \sqrt{\epsilon} N + \epsilon \left(iH_+ - S_+\right) + o(\epsilon)\right)$$

$$B_- = \frac{1}{\sqrt{2}} \left(\mathbb{I} - \sqrt{\epsilon} N + \epsilon \left(iH_- - S_-\right) + o(\epsilon)\right) \tag{8}$$

$$\text{with} \quad S_+ + S_- = N^\dagger N$$

Writing $H = \frac{H_+ + H_-}{2}$ and $dx^2 = \epsilon = dt$ one gets eventually:

Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\triangle\rho - \left(N\frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial x}N^\dagger\right) + i\,[H,\rho] + N^\dagger \rho N - \left\{\frac{NN^\dagger}{2}, \rho\right\} \quad (9)$$

Which shows a convective term similar to what one can see in the trivial model.

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$$\rho_{n+1} = \frac{B_+\rho_n B_+^\dagger}{tr(B_+\rho_n B_+^\dagger)}\mathbb{I}_{(\sigma_{n+1}=+1)} + \frac{B_-\rho_n B_-^\dagger}{tr(B_-\rho_n B_-^\dagger)}\mathbb{I}_{(\sigma_{n+1}=-1)} \quad (10)$$

$$X_{n+1} = X_n + \mathbb{I}_{(\sigma_{n+1}=+1)} - \mathbb{I}_{(\sigma_{n+1}=-1)} \quad (11)$$

This gives:

$$\begin{aligned}\rho_{n+1}-\rho_n =& \frac{1}{2}\left(\frac{B_+\rho_n B_+^\dagger}{tr(B_+\rho_n B_+^\dagger)}+\frac{B_-\rho_n B_-^\dagger}{tr(B_-\rho_n B_-^\dagger)}-2\rho_n\right)\\ &+\frac{1}{2}\left(\frac{B_+\rho_n B_+^\dagger}{tr(B_+\rho_n B_+^\dagger)}-\frac{B_-\rho_n B_-^\dagger}{tr(B_-\rho_n B_-^\dagger)}\right)(X_{n+1}-X_n)\end{aligned} \tag{12}$$

Then inserting the ϵ-development one gets:

$$\begin{aligned}\rho_{n+1}-\rho_n &= \epsilon\left(\frac{i}{2}[H,\rho_n]+N\rho_n N^\dagger-\frac{1}{2}\{N^\dagger N,\rho_n\}-2\Re(tr(N\rho_n))\left(N\rho_n+\rho_n N^\dagger\right)+4\Re(tr(N\rho_n))^2\rho_n\right)\\ &\quad+\sqrt{\epsilon}\left(N\rho_n+\rho_n N^\dagger-2\Re\,[tr(N\rho_n)]\,\rho_n\right)(X_{n+1}-X_n)\\ X_{n+1}-X_n &= \pi_{n+1}+2\sqrt{\epsilon}\Re\,[tr(N\rho_n)]\end{aligned} \tag{13}$$

Where $\pi_n = X_n - \mathbb{E}(X_n|\mathcal{F}_{n-1})$

Eventually with the notations:

$$H = \frac{H_+ + H_-}{2}$$
$$\mathcal{L}(\rho) = N\rho N^\dagger - \frac{1}{2}\{N^\dagger N, \rho\}$$
$$D(\rho) = N\rho + \rho N^\dagger - 2\Re[tr(N\rho)]\rho$$
$$U(\rho) = 2\Re[tr(N\rho)]$$

We get:

Stochastic differential equation

$$\begin{aligned} d\rho_t &= \{i[H, \rho_t] + \mathcal{L}(\rho_t)\} dt + D(\rho_t) dW_t \\ dX_t &= dW_t + U(\rho_t) dt \end{aligned} \tag{14}$$

A good consistency check is to go back to the Fokker-Planck equation using the equivalence of the two approaches:

$$\int dx \rho(x,t)|x\rangle\langle x| = \mathbb{E}[\rho_t|X_t\rangle\langle X_t|]$$

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$$\int dx \rho(x,t)|x\rangle\langle x| = \mathbb{E}[\rho_t |X_t\rangle\langle X_t|]$$

Both sides are distributions: make them act on $f^* g$ and differentiate. Then $\partial_t \rho(x,t)$ will simply be found by duality.

$$d(\int dx \hat{\rho}(x,t) f^*(x) g(x)) = \mathbb{E}[d(\rho_t \langle f|X_t\rangle\langle X_t|g\rangle)]$$

Then one just has to compute.

- By Itô's Lemma one has:

$$d(\langle f|X_t\rangle\langle X_t|g\rangle) = \partial(f^*g)dW_t + \left[U(\rho_t)\partial(f^*g) + \frac{1}{2}\partial^2(f^*g)\right] \quad (15)$$

- Then

$$\begin{aligned} d(\int dx\hat{\rho}(x,t)f^*(x)g(x)) &= \mathbb{E}[d(\rho_t)\langle f|X_t\rangle\langle X_t|g\rangle + \rho_t d(\langle f|X_t\rangle\langle X_t|g\rangle) + d(\rho_t)d(\langle f|X_t\rangle\langle X_t|g\rangle)] \\ &= \mathbb{E}(\{i[H,\rho_t] + \mathcal{L}(\rho_t)\}\, f^*(X_t)g(X_t) + \\ &\quad \rho_t\left[U(\rho_t)\partial(f^*g)(X_t) + \frac{1}{2}\partial^2(f^*g)(X_t)\right] + D(\rho_t)\partial(f^*g)\ \ )dt \end{aligned}$$

Eventually one finds, as expected:

$$\boxed{\frac{\partial \rho}{\partial t} = \frac{1}{2}\triangle\rho - \left(N\frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial x}N^\dagger\right) + i\,[H,\rho] + N^\dagger \rho N - \left\{\frac{NN^\dagger}{2},\rho\right\}} \tag{16}$$

Example: Let us go back to our example. We restrict ourselves to real matrices $iH = i\sigma_2$ and $N = a\sigma_3$. Hence ρ_t can be written the following way:

$$\rho_t = \frac{1}{2}(\mathbb{I} + q_1\sigma_1 + q_3\sigma_3)$$

with $q_1^2 + q_3^2 \leq 1$ The SDE gives:

$$\begin{aligned} dq_3 &= q_1 dt + 2a(1 - q_3^2) dW_t \\ dq_1 &= -2(q_3 + a^2 q_1) dt - 2a q_1 q_3 dW_t \end{aligned} \tag{17}$$

Using state purification one can reduce further the number of parameters to one angle θ which verifies the following SDE:

$$d\theta_t = -2\left(1 + a^2 \cos\theta_t \sin\theta_t\right) dt - \boxed{2a\sin\theta_t} dW_t \qquad (18)$$

Still not a canonical form to apply Kramers' results $\longrightarrow$ change of variable: $y_t = -|\log\tan\theta_t/2|$ to get a SDE of the form:

$$dy_t = -V'(y_t) + 2a dW_t$$

With $V(y) = -2\left(\pm\sinh(y) + 2a^2 \log\cosh(y)\right)$

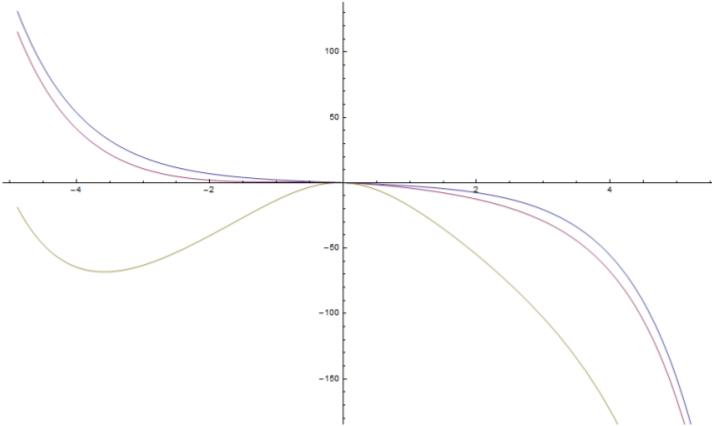

Figure : Potential $V(y) = -2\left(\pm\sinh(y) + 2a^2 \log\cosh(y)\right)$ for $a = 0.25$, $a = 1$ and $a = 3$

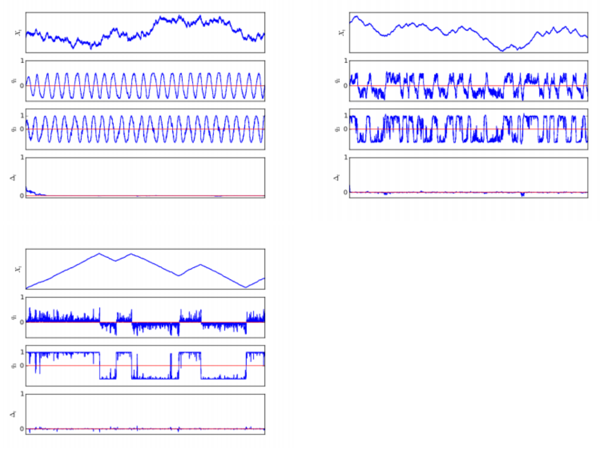

Figure : Illustration of the 3 regimes, for $a = 0.25$ *oscillating*, $a = 1$ *critical* and $a = 3.0$ *ballistic*

Eventually, one can use Kramers' results (basically the relation between kinetics and activation energy) for large a to get:

$$< \tau > \simeq e^{\frac{\Delta V}{4a^2}} \simeq a^2$$

Then the behavior for X_t is trivial because:

$$dX_t = dW_t + 2\Re\left(trN\rho_t\right)dt \simeq 2a(-1)^{N_t}dt$$

"Philosophical" summary of the talk

- Trajectories of OQRW are as simple as trajectories of classical processes and are therefore a complementary way to study OQRW, perhaps slightly underused.
- OQRW can give rise to very high diffusion constants with small noise terms.
- Counting processes can emerge from Gaussian fluctuations. There is no need to have a counting process in a Belavkin equation to see jumps.

Next on the theoretical side:

- Exhaustively explore the space of B matrices at least at the continuous limit.
- Investigate higher dimensions (both in internal space and position space).
- Include temperature in this description.
- Add disorder with random B matrices to see if Anderson localization breaks down with OQRW (idea of Alain Joye)

On the application side

- Find situations with unexpectedly high diffusion constants
- Find applications of the classical process to other fields (Biology for example).

For more information:

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Thank you for your attention.