Continuous tensor networks in d $\geq$ 1

possible approaches

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Why go to continuous TN

- Discrete TNs in $d \geq 2$ are hard to contract

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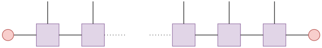
- Discrete TNs in $d \geq 2$ are hard to contract
- Model QFTs **directly**
- Why not?

Matrix product states

Definition of a MPS:

$$|A\rangle = \sum_{i_1, i_2, \cdots, i_n} \langle \mathsf{L}| A_{i_1} A_{i_2} \cdots A_{i_n} |\mathsf{R}\rangle \; |i_1, \cdots i_n\rangle$$

$=$ 

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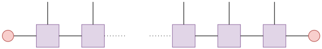
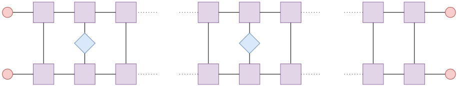
$=$ 

Operator expectation values:

$\langle A|\mathcal{O}(\ell_1)\mathcal{O}(\ell_2)|A\rangle =$ 

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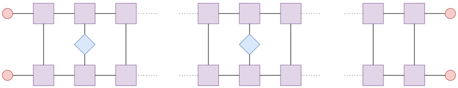
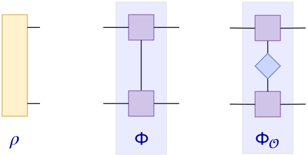


Evolution picture

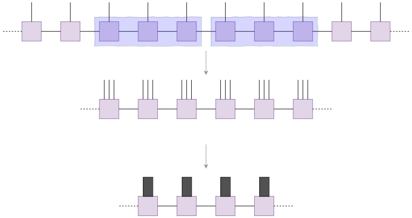


$$\langle A|\mathcal{O}(\ell_1)\mathcal{O}(\ell_2)|A\rangle = \mathrm{tr}\Big[\rho_R\, \Phi^{n_1}\cdot\Phi_{\mathcal{O}}\cdot\Phi^{n_2}\cdot\Phi_{\mathcal{O}}\cdot\Phi^{n_3}\,\rho_L\Big]$$

A can be seen as prescribing **dynamics** (or laws) rather than as a **state**

From MPS to cMPS

Zooming out



From MPS to cMPS

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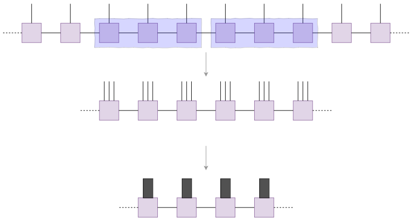

- the bond dimension stays fixed
- the physical dimension explodes $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \longrightarrow \mathscr{F}(L^2([x, x+dx]))$.

From MPS to cMPS

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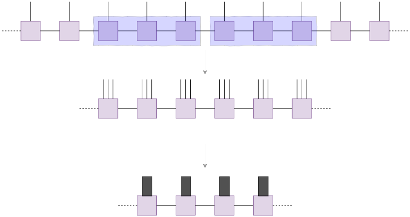
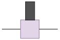


- the bond dimension stays fixed
- the physical dimension explodes $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \longrightarrow \mathscr{F}(L^2([x, x+dx]))$.
  $\Longrightarrow$ **Spins** become **fields**.
- a cMPS aims at describing  in continuous space

cMPS

Type of ansatz

- Matrices $A_{i_k}(x)$ where the index i_k corresponds to $a^{\dagger i_k}(x)|0\rangle$ in physical space.

Informal cMPS definition

$$
\begin{aligned}
A_0 &= \mathbb{1} + \varepsilon Q \\
A_1 &= \varepsilon R \\
A_2 &= \frac{(\varepsilon R)^2}{\sqrt{2}} \\
&\dots \\
A_n &= \frac{(\varepsilon R)^n}{\sqrt{n}} \\
&\dots
\end{aligned}
$$

so we go from ∞ to 2 matrices

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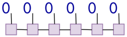
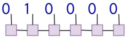
- Finite particle number



$\propto \quad 1$



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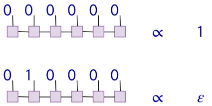
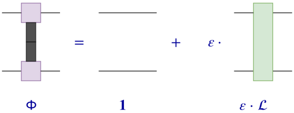


- Non trivial map



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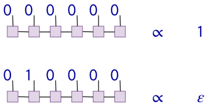

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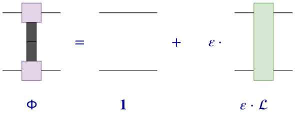

- Consistency

cMPS

Formal cMPS definition

$$|Q,R\rangle = \langle\omega_L|\mathcal{P}\exp\left\{\int_0^L dx\; Q\otimes\mathbb{1} + R\otimes a^\dagger(x)\right\}|\omega_R\rangle\;|0\rangle$$

Idea:

$$\begin{aligned} A(x) &\simeq A_0\mathbb{1} + A_1 a^\dagger(x) \\ &\simeq \mathbb{1}\otimes\mathbb{1} + \varepsilon Q\otimes\mathbb{1} + \varepsilon R\otimes a^\dagger(x) \\ &\simeq \exp\left[\varepsilon\left(Q\otimes\mathbb{1} + R\otimes a^\dagger(x)\right)\right] \end{aligned}$$

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-

Equivalent cMPS definition

$$|K,R\rangle = \langle\omega_L|\mathcal{P}\exp\left\{-i\int_0^L dx\ K\otimes\mathbb{1} + iR\otimes a^\dagger(x) - iR^\dagger a(x)\right\}|\omega_R\rangle\ |0\rangle$$

with $iK = Q + \frac{1}{2}R^\dagger R$.

cMPS

Field wave function

$$|\Psi\rangle = \sum_{n=0}^{+\infty} \int_{[0,L]^n} dx_1 \cdots dx_n \; \psi_n(x_1, \cdots, x_n) \; a^\dagger(x_1) \cdot a^\dagger(x_n) \; |0\rangle$$

cMPS

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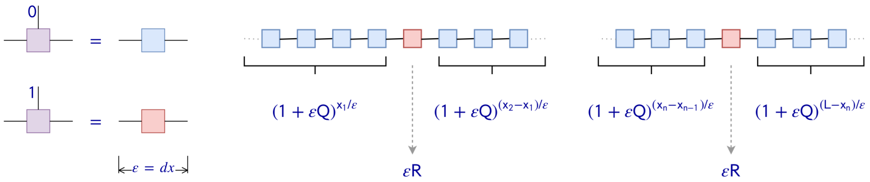
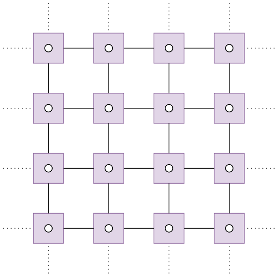
Field wave function for the cMPS

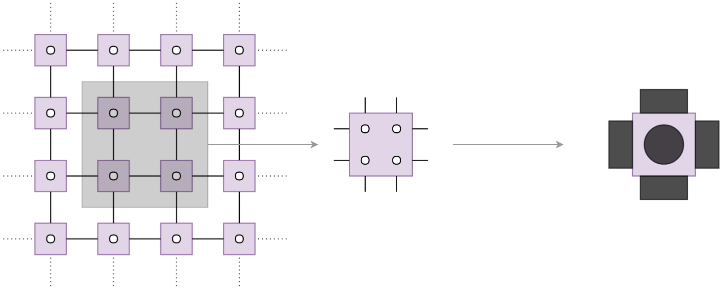


$$\psi_n(x_1, \cdots, x_n) dx_1 \cdots dx_n = e^{x_1 Q} \, R \, e^{(x_2 - x_1)Q} \cdots e^{(x_n - x_{n-1})Q} \, R \, e^{(L - x_n)Q} \underbrace{\varepsilon \cdot \varepsilon \cdots \varepsilon}_{dx_1 \cdots dx_n}$$

Tensor networks in 2d



Zooming out



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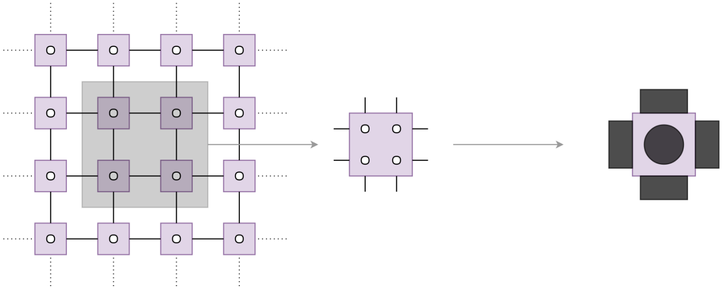
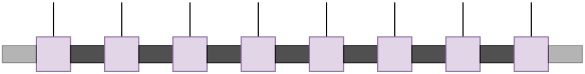


- The physical dimension becomes infinite
- **The bond dimension becomes infinite**

Do we need an ansatz with a field theory on the virtual space?

iMPS

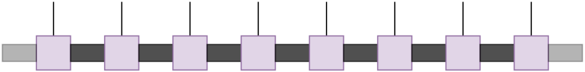


Take $A_i(x_i)$ an operator of a QFT

Naive example: $\hat{\Psi}_{0,1}$ is a spin $1/2$ field, $A_i(x_i) = \hat{\Psi}_i(x_i)$

$$|iMPS\rangle = \sum_{i_1,\cdots,i_n} \langle 0|\hat{\Psi}_{i_1}(x_1)\cdots\hat{\Psi}_{i_n}(x_n)|0\rangle\, |i_1,\cdots,i_n\rangle$$

iMPS



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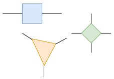
- wave function of something = correlation function of something else
- typically $A_i(x_i)$ a vertex operator of a CFT
- not dense in field states but analytic

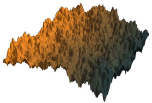
More traditional/naive continuous TNs

A **continuous TN** is an object that takes a (discrete) bunch of (discrete) tensors and spits a field state:

$$|A, B, \cdots\rangle = \sum\!\!\!\!\!\!\!\int \psi(x_1, \cdots, x_n)\; a^\dagger(x_1) \cdots a^\dagger(x_n)\, |0\rangle$$



$\longrightarrow$



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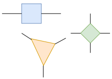

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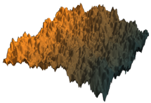
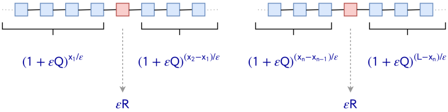


- **dense** in field states
- not **too** abstract
- maybe not in **1 to 1** with the discrete

A few options

Get inspiration from **cMPS**:

1. Extending the "wave function" picture $\to$ **random curves**

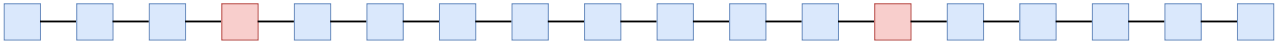


2. Extending the $\mathcal{P}$-ordered product $\to$ **path integral**

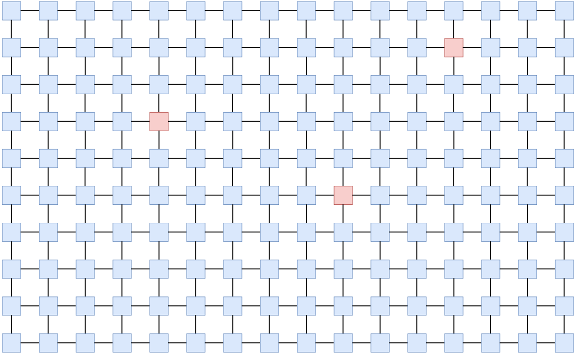
$$|K, R\rangle = \langle \omega_L | \mathcal{P} \exp \left\{ -i \int_0^L dx\ K \otimes \mathbb{1} + iR \otimes a^\dagger(x) - iR^\dagger a(x) \right\} |\omega_R\rangle\ |0\rangle$$

"Wave function" picture

For **cMPS**:



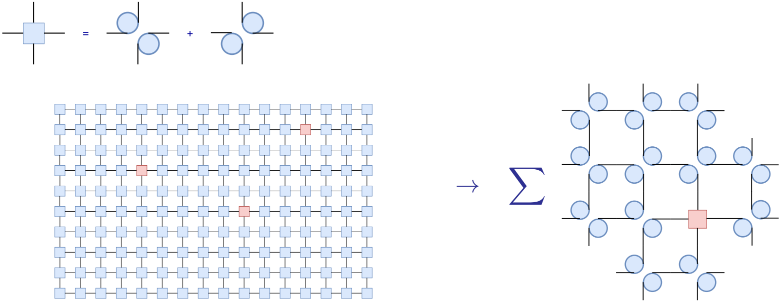
For **cPEPS**:



with $a^\dagger(x_i)$ and $\mathbb{1}$ on the physical space

“Wave function” picture

Take as the **ansatz**:



$\Longrightarrow$ Sum on self-avoiding loops with defects

“Wave function” picture

Along **a line**, just a product of matrices $\sim$ $\exp(\ell\,\hat{Q})$

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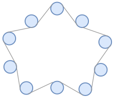
Along **a closed loop without defect**, a trace $\sim$ $\mathrm{tr}[\exp(\ell\,\hat{Q})]$



Take $\hat{Q} = i\hat{H}$, s.t. $\mathrm{tr}[\exp(\ell\,\hat{Q})] = 1$ to remove the contribution of loops without defects

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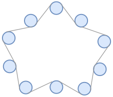
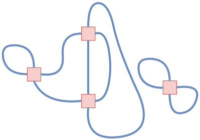


Take $\hat{Q} = i\hat{H}$, s.t. $\mathrm{tr}[\exp(\ell\,\hat{Q})] = 1$ to remove the contribution of loops without defects

Finally, the **wave function** reads as a sum of:



Each line of length ℓ represents the matrix $\exp[\ell\hat{Q}]$

Continuum limit and generalization

A few difficulties still:

- Self avoiding loops are not trivial to manage
- Model not manifestly isotropic

Continuum limit and generalization

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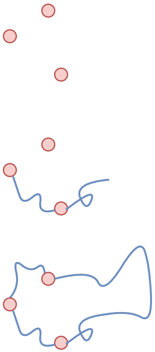
Tentative cTN definition

Main objects:

- A self-adjoint matrix H
- A fully symmetric n-tensor T
- A family of random curves $\mathscr{C}$ with Euclidean invariant probability distribution $\mathrm{d}\mu(\mathscr{C})$

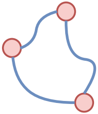
Construction:

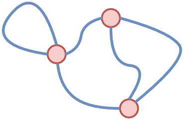
- Propagate a curve until nodes T are hit n times
- Contract the network, with each line $e^{i\ell H}$
- Sum $\int_{\mathscr{C}} \cdot \, \mathrm{d}\mu(\mathscr{C})$



Comments

- A cTN is the limit of a TN only in $d = 1$
- The connectivity of the graph is unrelated to the dimension





- In $2d$, lots of nice random curves: Brownian motion, SLE...
- Provides a natural deformation of conformal field theories
- Of course, very non-explicit so far

back to cMPS

Operator definition:

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Path integral representation of cMPS

$$|K,R\rangle = \int\mathcal{D}[\phi]e^{-S_{\text{kinetic}}(\phi,\phi^*)}\exp\left\{-i\int_0^L \phi_x^\dagger K\phi_x\;\mathbb{1} + i\phi_x^\dagger R\phi_x a^\dagger(x) - i\phi_x^\dagger R^\dagger\phi_x a(x)\right\}|0\rangle$$

with the Euclidean action:

$$S_{\text{kinetic}}(\phi,\phi^*) = i\int_0^L \phi_x^\dagger\partial_x\phi_x$$

Equivalently:

$$|K,R\rangle = \int \underbrace{\mathrm{d}\mu(\phi)}_{\text{measure}}\,\underbrace{\mathrm{e}^{-i\int_0^L \phi_x^\dagger K\phi_x}}_{\text{phase}}\;\underbrace{|\phi^\dagger R\phi\rangle}_{\text{coherent state}}$$

Path integral in $d \geq 2$

Path integral definition of cTN

Basic objects:

- n random fields ϕ_i with Euclidean invariant probability
- a self-adjoint $n \times n$ "**phase**" matrix K
- a generic $n \times n$ "**coherent amplitude**" matrix R

Construction:

$$|K, R\rangle = \mathbb{E}_\phi \left[\exp\left\{ -i \int_0^L \mathrm{d}^d x \, \phi_x^\dagger K \phi_x \right\} |\phi^\dagger R \phi\rangle \right]$$

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A cTN is obtained by "mixing" **statistical field theories**.

- Can be evaluated by Monte-Carlo
- What about CFTs?

Conclusion

- For a cTN, we may want **finite** or **infinite** bond dimension
- If finite, 2 "natural" **different** constructions:
  - with random curves
  - with random fields
- Carefully choosing the randomness complements the optimization of the tensor
- Nothing is done but it seems (at least mathematically) interesting