UNDERSTANDING JUMPS AND SPIKES IN CONTINUOUS QUANTUM TRAJECTORIES

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Laboratoire de Physique théorique, École Normale Supérieure Paris

MPQ theory seminar, February 2016

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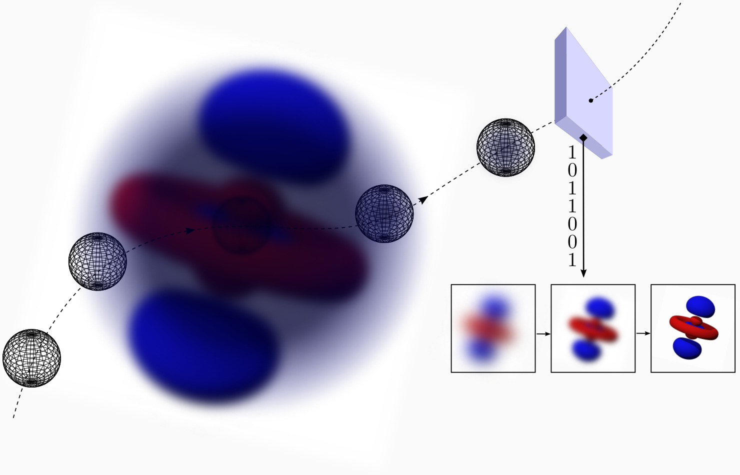

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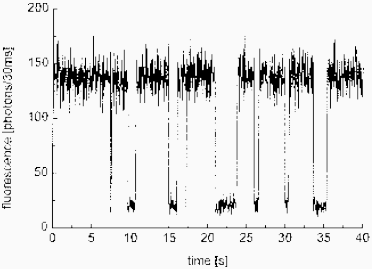
ABOUT

Work done with **Denis Bernard** and **Michel Bauer**.

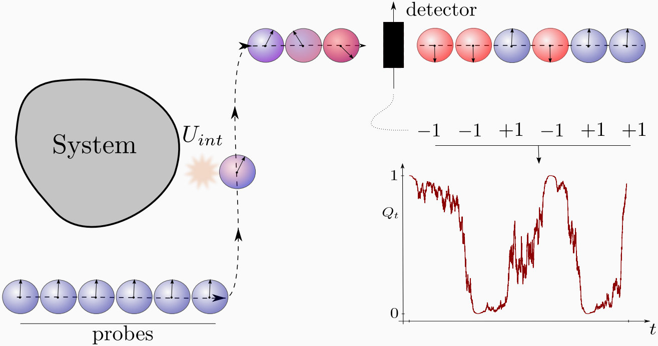
The objective is to understand the emergence of quantum jumps from a finer study of continuous measurements. See quantum jumps as the limit of some more detailed evolution. Possibly discover new phenomena.



OUTLINE

1. Continous measurements
2. Jumps
3. Spikes

How do you make a continuous measurement?



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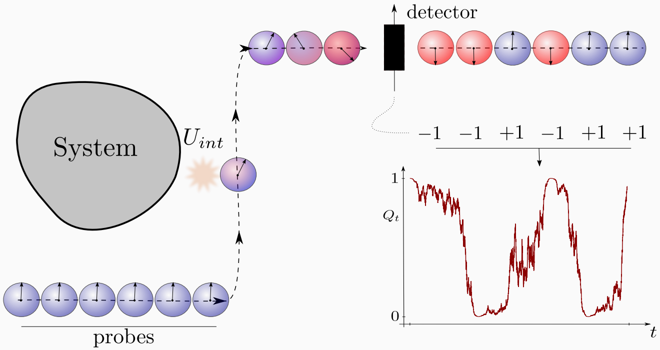

There are other ways of deriving the same results: weak coupling with infinite bosonic bath + unravelling, quantum noises, modified path integrals...

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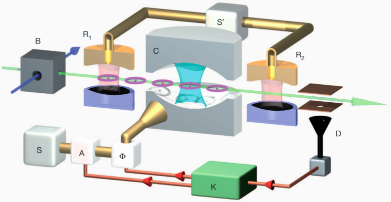
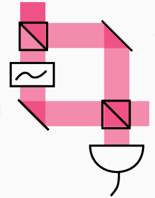
Ideal situations of application

- Discrete situations "a la Haroche", with actual repeated interactions

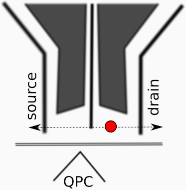


- "True" continuous measurement settings (homodyne detection in quantum optics)

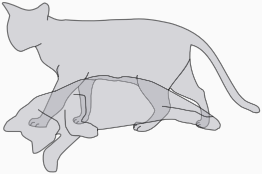


Other applications

- **Any** progressive measurement (e.g. quantum point contacts)



- Dynamical reduction models in foundations* (not today)



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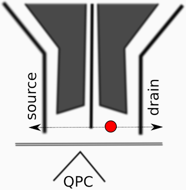

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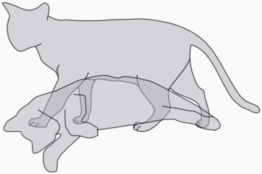

* actually they can always be formally written as the continuous measurement of something

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MODEL

System Hilbert space $\mathcal{H}_\mathrm{s}$, “probe” Hilbert space $\mathcal{H}_\mathrm{p} = \mathbb{C}^2$. The full density matrix is initially in a product state: $\rho = \rho_\mathrm{s} \otimes |+\rangle\langle+|$

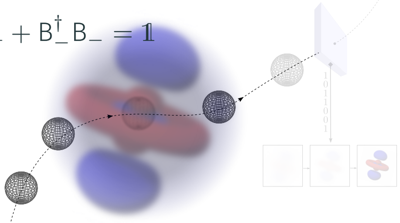
One weak measurement consists in:

1– **Unitary evolution** entangling the system and the probe:

$$\rho \to \mathrm{B}_+ \rho_\mathrm{s} \mathrm{B}_+^\dagger \otimes |+\rangle\langle+| \; + \; \mathrm{B}_- \rho_\mathrm{s} \mathrm{B}_-^\dagger \otimes |-\rangle\langle-|$$

$\sim$ to taking a picture of the particle but not looking at it yet.

Unitarity only implies: $\mathrm{B}_+^\dagger \mathrm{B}_+ + \mathrm{B}_-^\dagger \mathrm{B}_- = \mathbb{1}$



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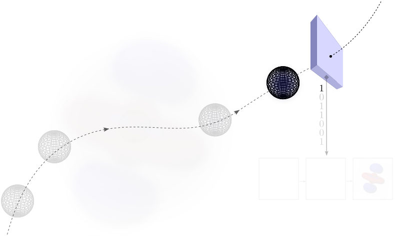
2– **Measurement** of the probe

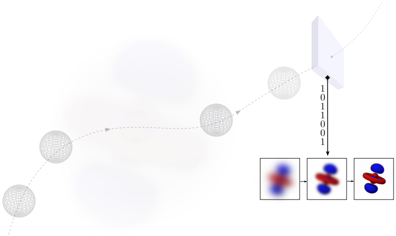
$$\rho \to \frac{B_\pm \rho_S B_\pm^\dagger \otimes |\pm\rangle\langle\pm|}{\mathrm{tr}(B_\pm \rho_S B_\pm^\dagger)} \quad \text{and result } \pm 1$$

$\sim$ to reading the picture and updating the probability



MODEL

3– Forgetting about the probe and taking a **new** one $|+\rangle\langle+|$ for the next iteration



Scaling

Develop B_+ and B_- in the vicinity of $\mathbb{1}/\sqrt{2}$ with the constraint:

$$B_+^\dagger B_+ + B_-^\dagger B_- = \mathbb{1}$$

Particular solution

$$B_\pm = \frac{1}{\sqrt{2}}\left[1 \pm \sqrt{\epsilon}N - \frac{\epsilon}{2}N^\dagger N + \mathcal{O}(\epsilon^{3/2})\right]$$

where N is just any matrix.

Next steps

- Compute $\mathrm{d}\rho(t) = \rho(t + \mathrm{d}t) - \rho(t)$ with $\mathrm{d}t = \epsilon$ explicitly (expand everything up to order dt).
- Separate the random part coming from the measurement in [average] + [noise with zero average] (Doob martingale decomposition)
- Notice that [noise with zero average] becomes white noise in the continuous limit

New equation:

$$\boxed{\mathrm{d}\rho_t = \mathscr{L}(\rho_t)\,\mathrm{d}t \;+\; \gamma\,\mathcal{D}[N](\rho_t)\,\mathrm{d}t \;+\; \sqrt{\gamma}\,\mathcal{H}[N](\rho_t)\,\mathrm{d}W_t} \tag{1}$$

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- $\mathscr{L}(\rho_t)$ your favorite **evolution** in the absence of measurement:

"unitary"
$$\mathscr{L}(\rho) = -i[H, \rho]$$

"thermal"
$$\mathscr{L}(\rho) = \Gamma_\uparrow\left(\sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho\}\right) + \Gamma_\downarrow\left(\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}\right)$$

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- $\mathcal{D}[N](\rho_t)$ the **dissipation** induced by measurement:

$$\mathcal{D}[N](\rho) = N\rho N^\dagger - \frac{1}{2}\{N^\dagger N, \rho\}$$

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- $\mathcal{H}[N](\rho_t)$ the stochastic **innovation** given by measurement

$$\mathcal{H}[N](\rho) = N\rho + \rho N^\dagger - \mathrm{tr}[(N + N^\dagger)\rho]\rho$$

This is not new

This kind of equations is known since the late eighties:

- in mathematical physics – Barchielli '82, Belavkin '89
- foundations – Diosi '88, Pearle, Gisin
- in quantum optics + feedback – Milburn & Wiseman '94

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- It is possible to talk about the strength/intensity/rate of a measurement.
- Continuous measurements can model an actual measurement situation or some elusive fundamental dynamical collapse.
- Gives you new equations (like Schrödinger or Lindblad) to play with.

What happens when measurement dominates the evolution ?

[typically when $\gamma \gg \Gamma_{\uparrow/\downarrow}$]

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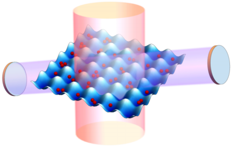
Question which has recently attracted some interest:

- Quasi-Zeno dynamics in condensed matter –Elliott & Vedral 1601.06624, Kozlowski et al. 1510.04857–



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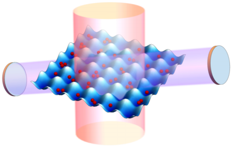
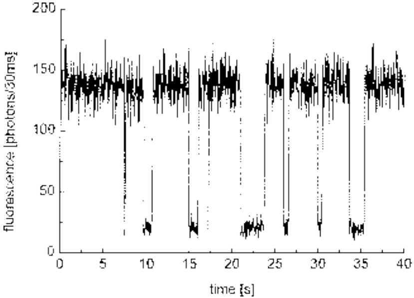


- Analysis of quantum jumps in simple situations with least action principle –Jordan & co, Sidiqqi & co 1305.5201, 1403.4992–

We imagine that the system state will collapse on measurement eigenvectors and possibly jump between them



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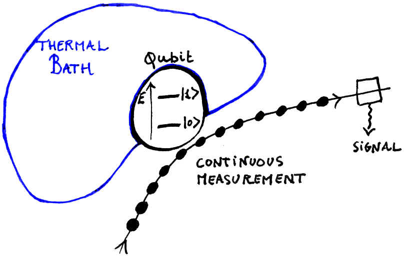
Coupling to a bath + energy measurement

$$\rightarrow \mathscr{L}(\rho) = \Gamma_{\uparrow}\left(\sigma_{+}\rho\sigma_{-} - \frac{1}{2}\{\sigma_{-}\sigma_{+}, \rho\}\right) + \Gamma_{\downarrow}\left(\sigma_{-}\rho\sigma_{+} - \frac{1}{2}\{\sigma_{+}\sigma_{-}, \rho\}\right)$$

$$\rightarrow N = \sigma_z$$



What follow is generic and can be obtained for $\mathscr{L}(\rho) = -i[H, \rho]$, i.e. for a fully coherent evolution preserving **pure states**.

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Equation after simplifications

Let us write $Q_t = \langle 0 | \rho_t | 0 \rangle$. The equation $d\rho_t = ...$ gives :

$$dQ_t = [\underbrace{-\Gamma_\uparrow Q_t + \Gamma_\downarrow (1 - Q_t)}_{\text{thermalbath}}]dt + \underbrace{\sqrt{\gamma} Q_t (1 - Q_t)}_{\text{mesurement}} dW_t$$

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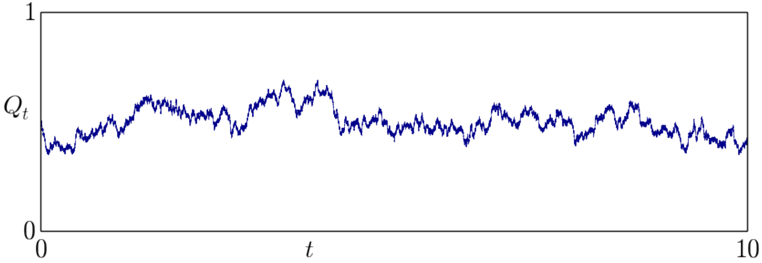
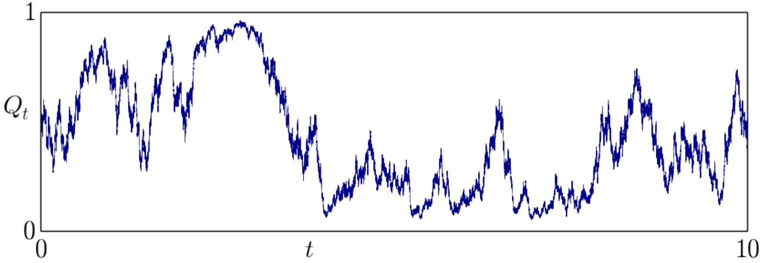
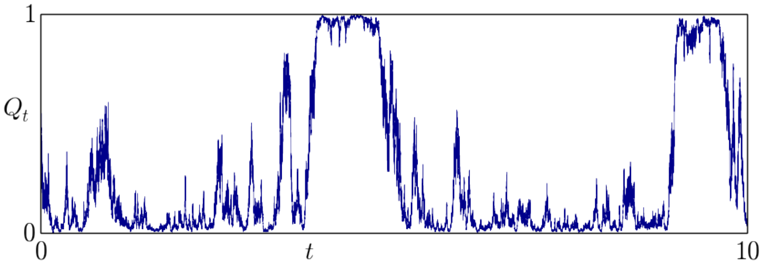
For simulations I fix $\Gamma_\uparrow = \Gamma_\downarrow = 1$, that is $T = +\infty$ but the results are generic. In the absence of measurements $Q \to 1/2$.



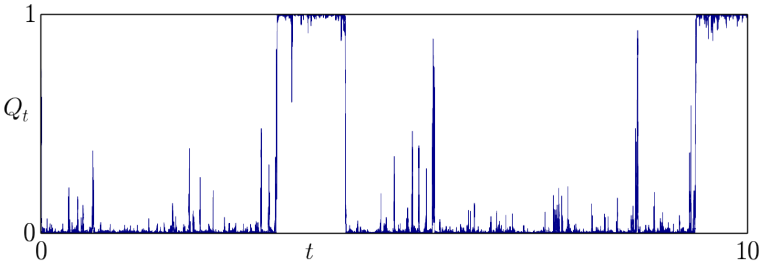
$\gamma = 0.1$



$\gamma = 1.0$



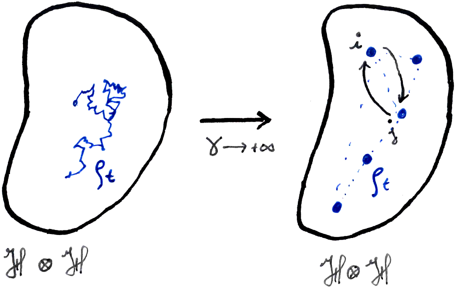
$\gamma = 10$



$\gamma = +\infty$

“JUMP THEOREM”

Qualitatively



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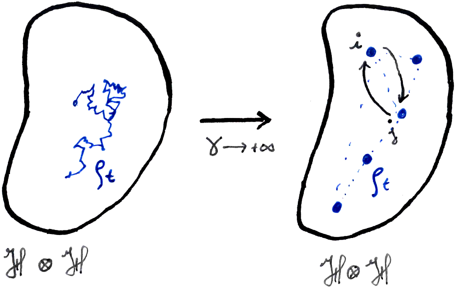


Quantitatively

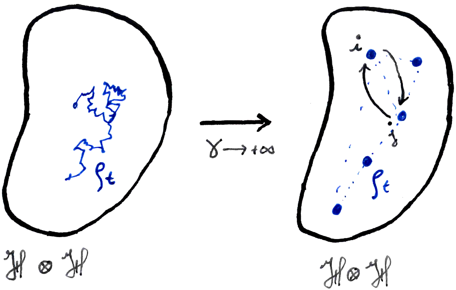
$$m_{i\to j} \propto \frac{[\text{coeffs. of H}]^2}{\gamma\,[\text{coeffs. of N}]^2} + [\text{Coeffs. dissipative part of evol.}]$$

Zeno−renormalized

no Zeno

"JUMP THEOREM"

Qualitatively



Quantitatively

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Notably, the eigenvalues of the measured operator matter!

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Crude idea of the proof

Not completely standard because strong noise limit $\longrightarrow$ “perturb” around the pure measurement situation

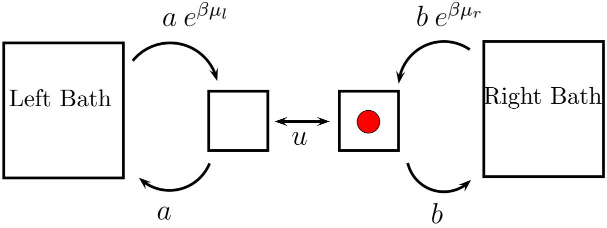
- Consider the probability kernel $K_t(\rho_0, d\rho)$ to go from a given density matrix ρ_0 to another density matrix ρ, up to $d\rho$, after a time t.
- Write its Kolmogorov equation $\partial_t K = K\mathfrak{D}$ where $\mathfrak{D}$ can be expanded in:

$$\mathfrak{D} = \gamma\mathfrak{D}_2 + \mathfrak{D}_0$$

- Compute the eigenvectors of $\mathfrak{D}_2$ (invariant measures) and perturbatively expand $K_t = e^{t\gamma\mathfrak{D}_2 + t\mathfrak{D}_0}$

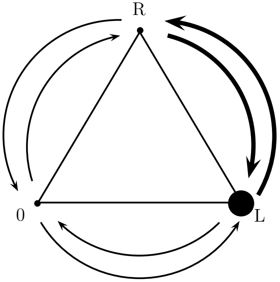
Control

Exploit the different behavior of unitary quantum jumps and thermal quantum jumps with respect to the Zeno effect to control systems



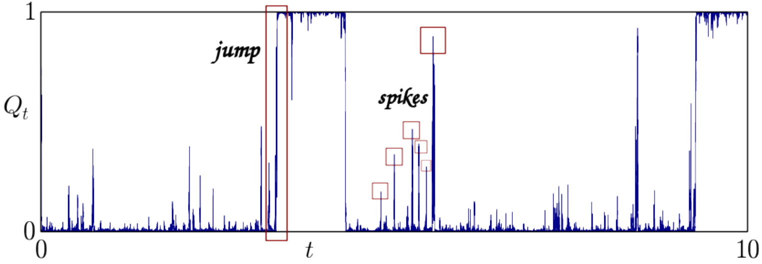
3 states :$|L\rangle$, $|R\rangle$, $|0\rangle$. Unitary coupling (tunneling) between L and R, dissipative between 0 and L and 0 and R.

Control

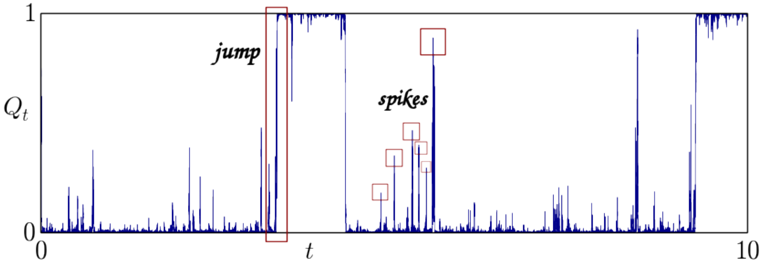


Maxwell Daemon using only the measurement strength!

GO BACK TO THE NUMERICS

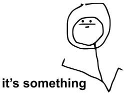


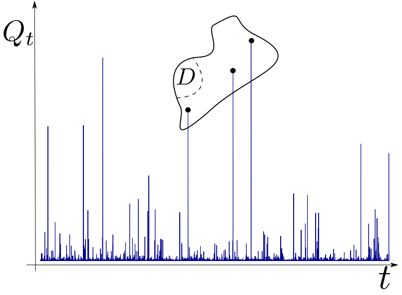
$\gamma = +\infty$



$\gamma = +\infty$

For the **spikes** I expect what I say to be generic but can only prove it in 2d.





The number n of spikes in the domain D is a Poisson process of intensity $\mu = \int_D d\nu$ with :

$$d\nu = \frac{\Gamma_\uparrow}{Q^2}\, dQdt$$

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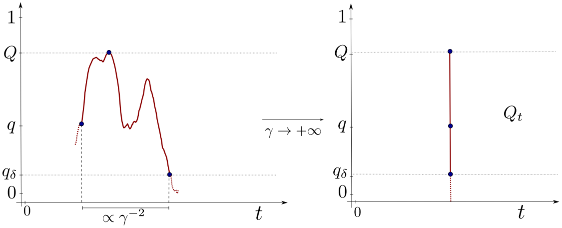
Crude idea of the proof



- Out of the boundary: almost pure measurement, probabilities can be computed from the martingale property.
- Compute the probability of the max of an excursion.

Summary

- Continuous measurement $\underset{\gamma\to+\infty}{\sim}$ Repeated projective measurements
  - $\rightarrow$ quantum jumps

Summary

- Continuous measurement $\underset{\gamma \to +\infty}{\sim}$ Repeated projective measurements
  - $\to$ quantum jumps
- Continuous measurement $\underset{\gamma \to +\infty}{\neq}$ Repeated projective measurements
  - $\to$ Jump rates depend on the eigenvalues of the measured operator
  - $\to$ Spikes do not vanish in the limit

What next

Directly related questions:

- Measure operators with continuous spectrum [explored in a special case by Bassi & Dürr '07 in foundations]
- Describe spikes in higher dimension
- Prove that the results hold for repeated weak measurements (not necessarily continuous)

What next

More broadly:

- Apply to systems with topological properties (robustness, reading the state)
- Explore the phase transitions in continuously monitored extended quantum systems (say spin chains)
- Study stochastic thermodynamics in the quantum regime
- Study optimal information extraction in finite time
- Study optimal control with finite information flow

- for a pedestrian derivation of the formalism 1312.1600
- for jumps 1410.7231
- for spikes 1510.01232 & 1512.02861
- for an application to control 1404.7391
- for an application in foundations 1509.08705
- for an application to quantum info 1511.06555

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General solution

$$B_{\pm} = \frac{1}{\sqrt{2}}\left[1 \pm \sqrt{\epsilon}N_{\pm} - \epsilon\left(\pm M_{\pm} + \frac{1}{2}N^{\dagger}_{\pm}N_{\pm}\right) + \mathcal{O}(\epsilon^{3/2})\right]$$

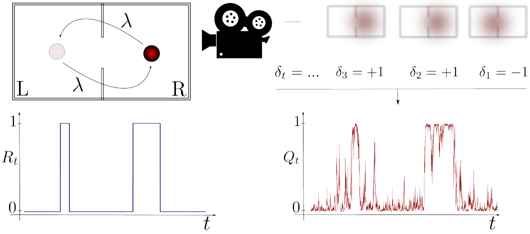
with $\Re e(N_+) = \Re e(N_-)$ and $\Re e(M_+) = \Re e(M_-)$

See e.g. `arXiv:1303.6658` or `arXiv:1312.1600`

ARE SPIKES «REAL», ARE SPIKES «QUANTUM»?

$\longrightarrow$ **Subtle question** a bit related to foundations and which depends on what we mean by real and quantum

- It is possible to build a purely classical model with spikes (hidden markov model)



ARE SPIKES «REAL», ARE SPIKES «QUANTUM»?

- Spikes disappear with forward backward filtering (Past Quantum State)

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- Spikes operationally exist in the sense that when using the quantum state to do feedback, spikes modify naive expectations.

Spikes can be seen/understood with a time reparametrisation $\rightarrow$ new effective time τ.

In the discrete case:

$$\Delta\tau_n = \mathrm{tr}\left[(\rho_{n+1} - \rho_n)^2\right] \Delta t_n$$

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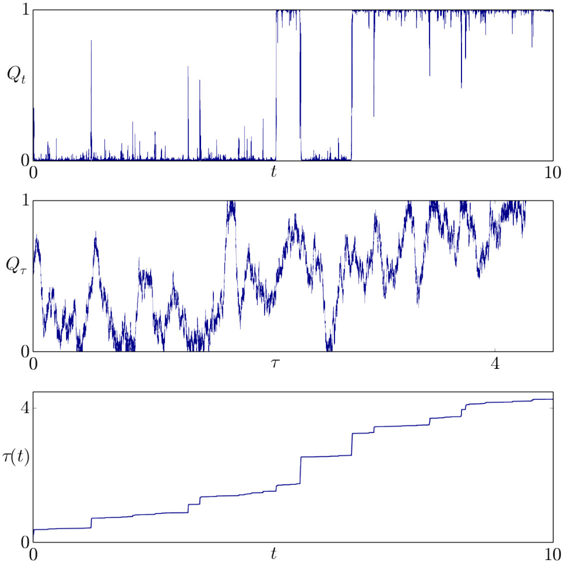
In the continuum:

$$d\tau = \mathrm{tr}\left[(d\rho_t)^2\right]$$



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Theorem

When $\gamma \to +\infty$, Q_τ becomes a **Brownian motion** reflected in 0 and in 1.

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- **jumps** correspond to transitions $0 \to 1$ and $1 \to 0$.
- **spikes** correspond to transitions $0 \to 0$ et $1 \to 1$.

Qubit in an external field

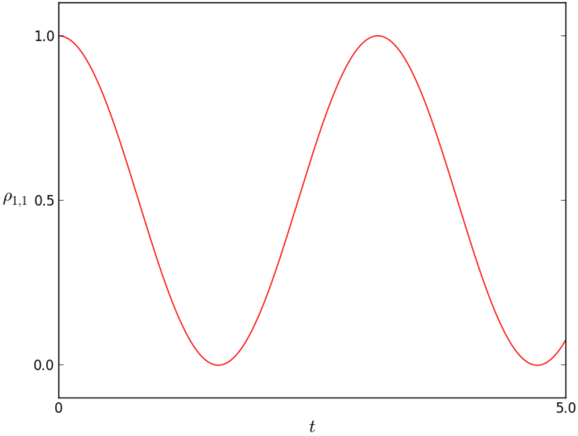
Consider a two level system (a qubit) with Hamiltonian $H = \frac{\omega}{2}\sigma_x$ with σ_z continuously monitored at a rate γ.

The evolution is given by the stochastic master equation:

$$d\rho_t = -i\frac{\omega}{2}[\sigma_x, \rho_t]dt + \underbrace{\gamma L_{\sigma_z}(\rho_t)dt + \sqrt{\gamma}D_{\sigma_z}(\rho_t)dW_t}_{\text{same measurement as before}}$$

We will look at $\langle+|\,\rho_t\,|+\rangle_z$, i.e. at the probabilities in the eigenbasis of the measurement.

Results



Without measurement $\gamma = 0.0$

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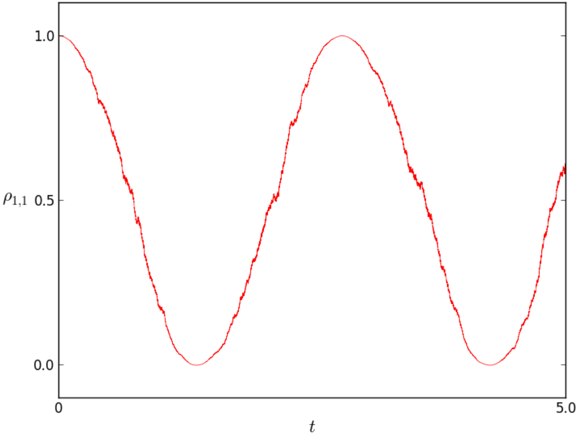
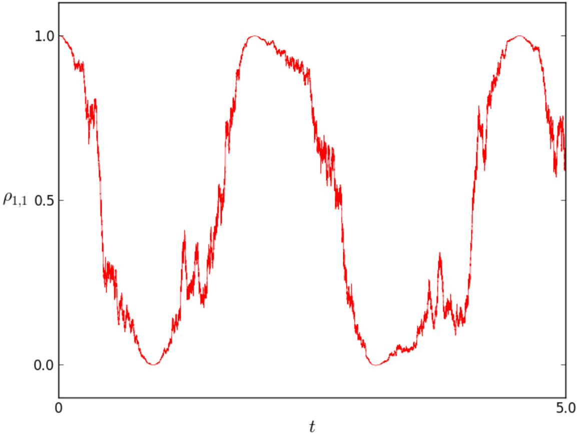
Results



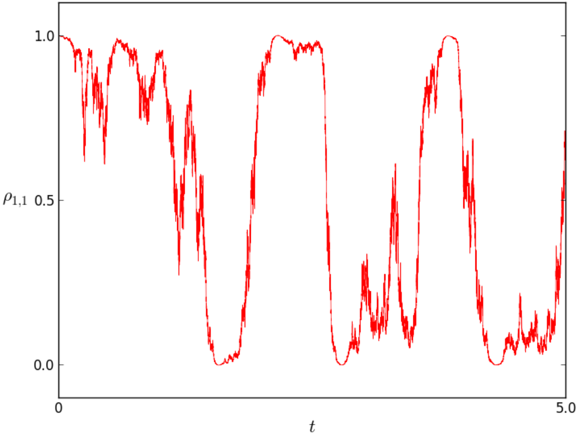
$\gamma = 0.1$

Results



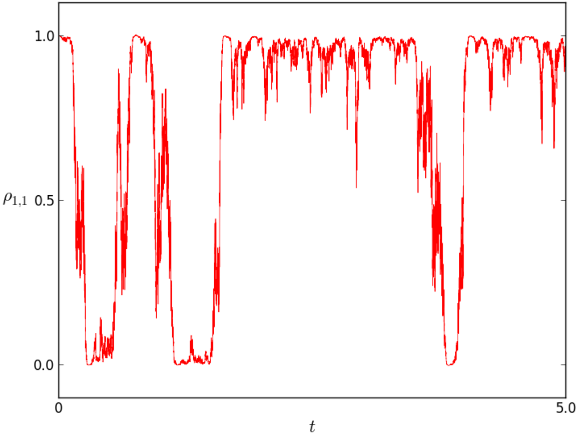
$\gamma = 0.5$

Results



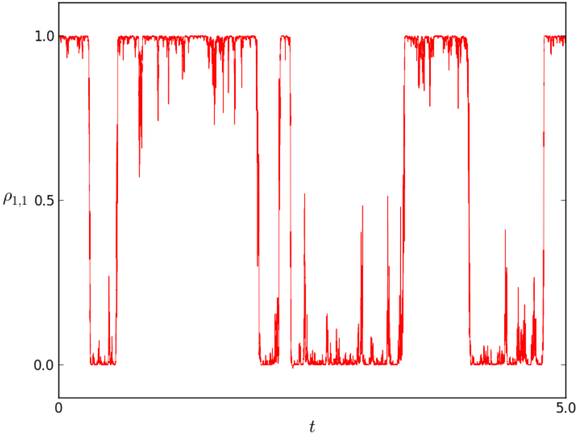
$\gamma = 1.0$

Results



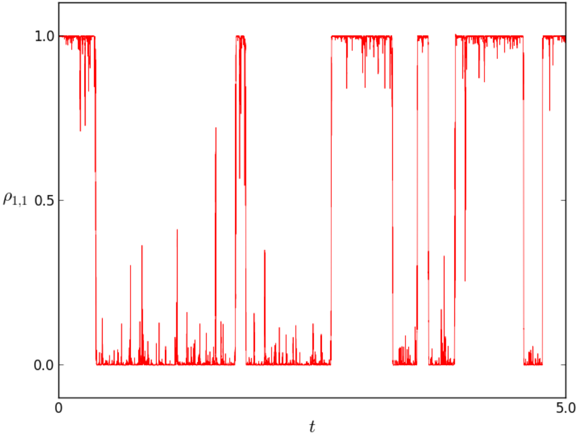
$\gamma = 2.0$

Results



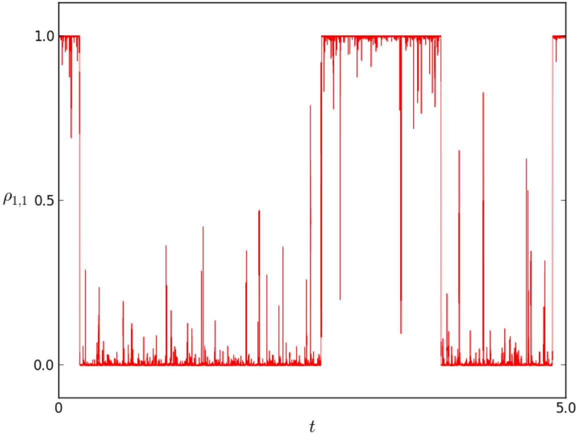
$\gamma = 5.0$

Results



$\gamma = 10$

Results



$\gamma = 20$

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Disclaimer

Actually, I had to cheat a bit and take $\omega \propto \gamma$ for the previous plots to counter the **Zeno effect**.