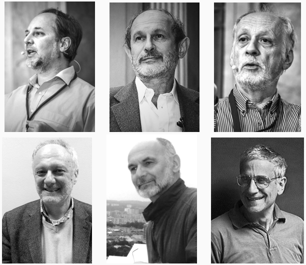
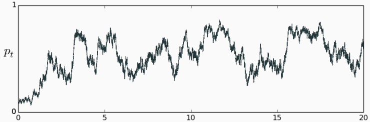
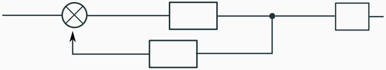
Stochastic calculus tools for quantum optics

PART II: Quantum trajectories and feedback







Quantum Optics Seminar

September 29, 2016

Antoine Tilloy, MPQ-theory

MPQ

THE STORY SO FAR

We gave a **precise** meaning to **Langevin** equations:

$$\dot{X}_t = \mu(X_t, t) + \sigma(X_t, t)\, \underbrace{\eta_t}_{\text{white noise}}$$

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which is compact a **notation** for:

$$X_t = \int^t \mu(X_t, t)\,\mathrm{d}t + \underbrace{\int^t \sigma(X_t, t)\,\mathrm{d}W_t}_{\text{Itô integral}}$$

Main finding:

We discovered that differentiation was **vicious**.

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Itô's lemma

Let $f \in \mathcal{C}^2$ and X_t an Itô process s.t.:

$$\mathrm{d}X_t = \mu(X_t, t)\,\mathrm{d}t + \sigma(X_t, t)\,\mathrm{d}W_t,$$

then:

$$\mathrm{d}f(X_t, t) = \underbrace{\frac{\partial f}{\partial t}\,\mathrm{d}t + \frac{\partial f}{\partial X}\,\mathrm{d}X_t}_{\text{trivial}} + \underbrace{\frac{1}{2}\frac{\partial^2 f}{\partial X^2}\,\sigma(X_t, t)^2\,\mathrm{d}t}_{\text{Itô correction}}$$

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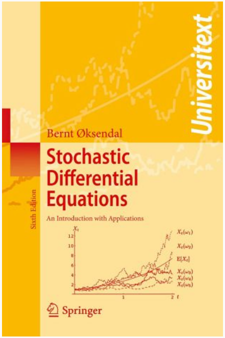
2. Construct the Brownian motion in 1D (Wiener process)
3. Construct an integral with respect to the Brownian motion
4. Show that it extends to all continuous random processes
5. Show that differentiation rules are changed

WHAT HAPPENED BEFORE

A **reference** for last time:

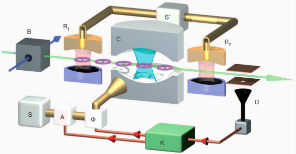
Very nice and easy book to learn Itô calculus $\longrightarrow$



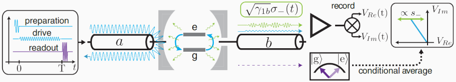
Bernt Øksendal

WHY DO WE NEED STOCHASTIC CALCULUS AT ALL

It is now possible to sequentially or continuously measure the **same** quantum system and implement a feedback depending on the results.



Discrete situation: experiment of the group of Serge Haroche, Gleyzes et al. Nature **446**, 297-300 (2007)

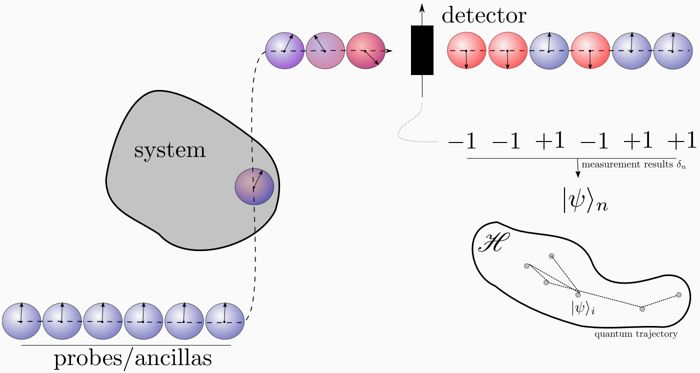


Continuous situation: experiment of the group of Benjamin Huard, Campagne-Ibarcq et al. Phys. Rev. Lett. **112**, 180402 (2014)

Master equations $\partial_t \rho_t = \mathscr{L}(\rho_t)$ are not enough, the state itself is random $\Rightarrow \partial_t \rho_t = \mathscr{L}(\rho_t) + \text{noise}\ (+\textbf{feedback})$

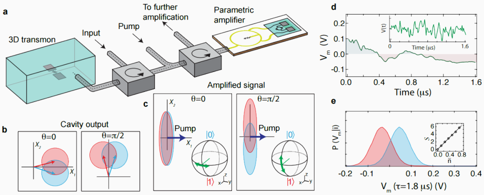
CONTINUOUS MEASUREMENT

REPEATED INTERACTION SCHEMES



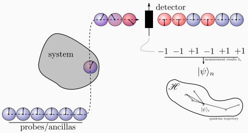
REPEATED INTERACTION SCHEMES

Implementation:



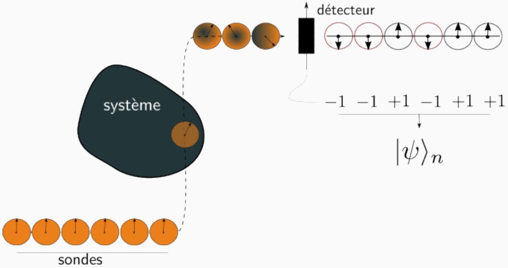
Experimental setup of the group of Irfan Siddiqi at Berkeley, Nature **502**, 211 (2013)

⟶ idealization



Situation considered

- System, $|\psi\rangle \in \mathscr{H}_s$
- Probe, $\mathscr{H}_p = \mathbb{C}^2$
- Unitary interaction
- Measurement of σ_z on the probe



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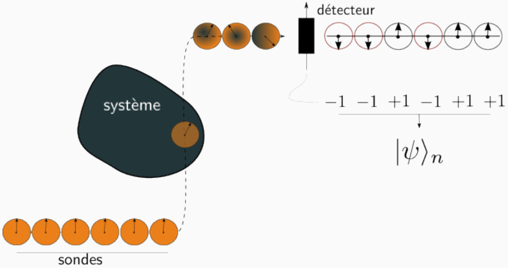
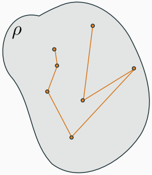


$$|\psi\rangle_n \otimes |+\rangle_x \xrightarrow{\text{interaction}} \hat{\Omega}_+ |\psi\rangle_n \otimes |+\rangle_z + \hat{\Omega}_- |\psi\rangle_n \otimes |-\rangle_z$$

$$\xrightarrow{\text{mesurement}} |\psi\rangle_{n+1} = \frac{\Omega_\pm |\psi\rangle_n}{\sqrt{\langle\psi| \Omega_\pm^\dagger \Omega_\pm |\psi\rangle_n}}$$

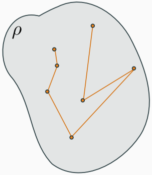
with the **only** constraint:

$$\Omega_+^\dagger \Omega_+ + \Omega_-^\dagger \Omega_- = \mathbb{1}$$



Discrete quantum trajectories

A sequence $|\psi\rangle_n$ or ρ_n (random) and the corresponding measurement results $\delta_n = \pm 1$.



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$\Rightarrow$ Make the interactions soft and frequent:

$$\Omega_\pm = \frac{1}{\sqrt{2}}\left(\mathbb{1} \pm \mathcal{O}\,\varepsilon + \#\,\varepsilon^2 + \cdots\right)$$

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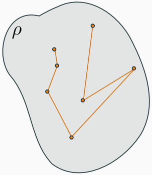

$\Rightarrow$ Make the interactions soft and frequent:

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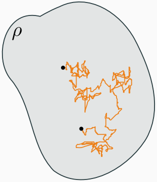
Continuous quantum trajectories

A continuous process $|\psi\rangle_t$ or ρ_t (random) and the corresponding measurement signal y_t:

$$y_t \propto \sqrt{\Delta t}\sum_{n=1}^{t/\Delta t} \delta_n$$



RESULT

Stochastic master equation (∼ 1987)

State (density matrix or pure state):

$$\mathrm{d}\rho_t = \mathscr{L}(\rho_t)\,\mathrm{d}t + \gamma\,\mathcal{D}[\mathcal{O}](\rho_t)\,\mathrm{d}t + \sqrt{\gamma}\,\mathcal{H}[\mathcal{O}](\rho_t)\,\mathrm{d}W_t$$

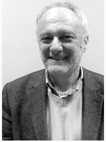
Signal:

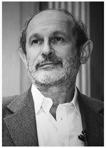
$$\mathrm{d}y_t = \sqrt{\gamma}\,\mathrm{tr}\left[\left(\mathcal{O} + \mathcal{O}^\dagger\right)\rho_t\right]\,\mathrm{d}t + \mathrm{d}W_t$$

with:

- $\mathcal{D}[\mathcal{O}](\rho) = \mathcal{O}\rho\mathcal{O}^\dagger - \frac{1}{2}\left(\mathcal{O}^\dagger\mathcal{O}\rho + \rho\mathcal{O}^\dagger\mathcal{O}\right)$
  «decoherence and dissipation»
- $\mathcal{H}[\mathcal{O}](\rho) = \mathcal{O}\rho + \rho\mathcal{O}^\dagger - \mathrm{tr}\left[\left(\mathcal{O} + \mathcal{O}^\dagger\right)\rho\right]\rho$
  «aquisition of information»
- $\frac{\mathrm{d}W_t}{\mathrm{d}t}$ white noise


V. Belavkin


A. Barchielli


L. Diósi

Real trajectories, different (in spirit) from Dalibard-Castin-Mølmer used for Monte Carlo with jumps

VOLUME 68, NUMBER 5 PHYSICAL REVIEW LETTERS 3 FEBRUARY 1992

Wave-Function Approach to Dissipative Processes in Quantum Optics

Jean Dalibard and Yvan Castin
Laboratoire de Spectroscopie Hertzienne de l'Ecole Normale Supérieure, 24 rue Lhomond, F-75231 Paris CEDEX 05, France

Klaus Mølmer
Institute of Physics and Astronomy, University of Aarhus, DK-8000 Aarhus C, Denmark
(Received 15 October 1991)

A novel treatment of dissipation of energy from a "small" quantum system to a reservoir is presented. We replace the usual master equation for the small-system density matrix by a wave function evolution including a stochastic element. This wave-function approach provides new insight and it allows calculations on problems which would otherwise be exceedingly complicated. The approach is applied here to a two- or three-level atom coupled to a laser field and to the vacuum modes of the quantized electromagnetic field.

PACS numbers: 42.50.−p, 32.80.−t

or from Gisin-Percival used for diffusive Monte Carlo

J. Phys. A: Math. Gen. **25** (1992) 5677–5691. Printed in the UK

The quantum-state diffusion model applied to open systems

Nicolas Gisin† and Ian C Percival‡
† Group of Applied Physics, University of Geneva, 1211 Geneva 4, Switzerland
‡ Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2BZ, UK, and School of Mathematical Sciences, Queen Mary and Westfield College, University of London, Mile End Road, London E1 4NS, UK

Received 8 May 1992

Abstract. A model of a quantum system interacting with its environment is proposed in which the system is represented by a state vector that satisfies a stochastic differential equation, derived from a density operator equation such as the Bloch equation, and consistent with it. The advantages of the numerical solution of these equations over the direct numerical solution of the density operator equations are described. The method is applied to the nonlinear absorber, cascades of quantum transitions, second-harmonic generation and a measurement reduction process. The model provides graphic illustrations of these processes, with statistical fluctuations that mimic those of experiments. The stochastic differential equations originated from studies of the measurement problem in the foundations of quantum mechanics. The model is compared with the quantum-jump model of Dalibard, Carmichael and others, which orginated among experimenters looking for intuitive pictures and rules of computation.

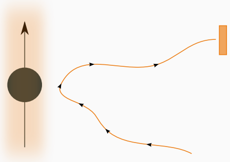
Situation considered

"Pure" continuous measurement of a qubit

Qubit $\Rightarrow \mathscr{H} = \mathbb{C}^2$ so $\rho_t = \begin{pmatrix} p_t & u_t \\ u_t^* & 1 - p_t \end{pmatrix}$

Continuous energy measurement, i.e.

$\mathcal{O} = \sigma_z \propto H$



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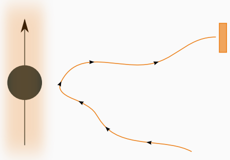

Equation for the population

$$\mathrm{d}p_t = \sqrt{\gamma}\, p_t(1-p_t)\,\mathrm{d}\,W_t$$

EXAMPLE

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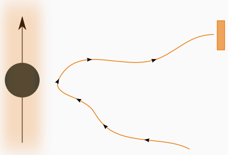


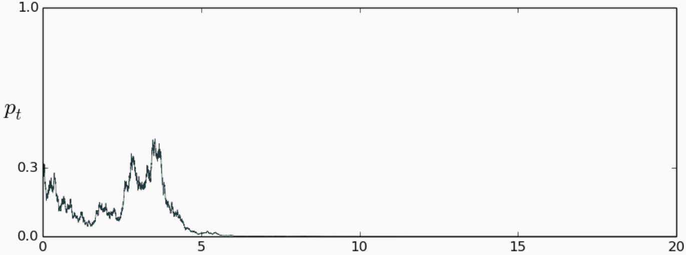
Equation for the population

$$\mathrm{d}p_t = \sqrt{\gamma}\, p_t(1 - p_t)\, \mathrm{d}W_t$$

Equation for the phase

$$\mathrm{d}u_t = -\frac{\gamma}{8} u_t\, \mathrm{d}t + \frac{\sqrt{\gamma}}{2}(2p_t - 1)\, \mathrm{d}W_t$$

"Pure" continuous measurement of a qubit



EXAMPLE

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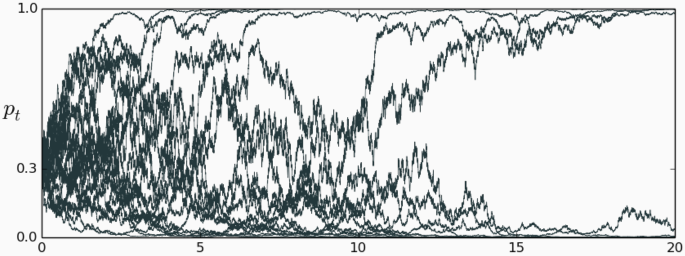

How do we see that the **Born** rule works from

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but $\mathbb{E}[p_\infty] = \mathbb{P}[p_t \to 1] \cdot 1 + \mathbb{P}[p_t \to 0] \cdot 0 = \mathbb{P}[p_t \to 1]$

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but $\mathbb{E}[p_\infty] = \mathbb{P}[p_t \to 1] \cdot 1 + \mathbb{P}[p_t \to 0] \cdot 0 = \mathbb{P}[p_t \to 1]$

so finally:

$$\mathbb{P}[p_t \to 1] = p_0$$

LOCAL CONCLUSION

→ Collapse now has a timescale γ^{-1}

→ The Born rule stays valid

→ The trajectory is real

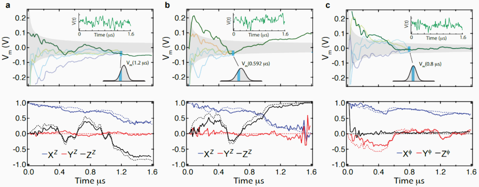

FIG. 3: Quantum trajectories. **a,b** Individual measurement traces obtained for Z-measurements with $\bar{n} = 0.4$. The top panel displays $V_m(\tau)$ as a green line, with the inset displaying the instantaneous measurement voltage. The gray region indicates the standard deviation of the distribution of measurement values. Measurement traces that converge to an integrated value within the blue matching window are used to tomographically reconstruct the trajectory at that time point. A few different measurement traces that contribute to the reconstruction at 1.2 μs (**a**) and 0.592 μs (**b**) are indicated in pastel colors. The lower insets indicate the distribution of measurement values with the matching window indicated in blue. Quantum trajectories obtained from analysis of the measurement signal are shown as dashed lines in the lower panel. Solid lines indicate the tomographically reconstructed quantum trajectory based on the ensemble of measurements that are within the matching window of the original measurement signal. **c** Individual measurement traces and associated quantum trajectory obtained for a ϕ-measurement with $\bar{n} = 0.4$.

Quantum trajectories from the group of Irfan Siddiqi at Berkeley, Nature **502**, 211 (2013)

How fast do we purify?

ONE EXAMPLE OF ITÔ WITHOUT FEEDBACK

How fast do we purify?

Look at $\Delta_t = \sqrt{\det(\rho_t)} = \sqrt{p_t(1 - p_t)}$ and compute $\mathrm{d}\Delta_t$

$$\mathrm{d}\Delta_t = -\underbrace{\frac{\gamma}{8}\Delta_t\,\mathrm{d}t}_{\text{Itô correction}} + \frac{1}{2}\sqrt{\gamma\Delta_t}(1 - 2p_t)\,\mathrm{d}W_t$$

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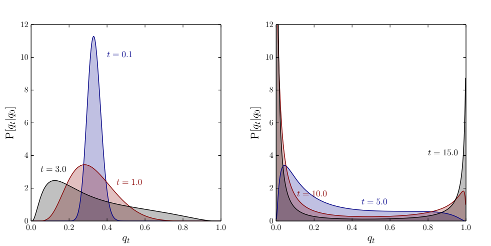
Let us look at the average $\bar{\Delta}_t = \mathbb{E}[\Delta_t]$

$$\frac{\mathrm{d}\bar{\Delta}_t}{\mathrm{d}t} = -\frac{\gamma}{8}\bar{\Delta}_t \quad \Rightarrow \quad \bar{\Delta}_t = \bar{\Delta}_0\, e^{-\frac{t}{8\gamma}}$$

OTHER WAYS?

$$
\begin{aligned}
\mathrm{d}\mathbb{P}[p_t = p | p_0] = & \frac{2p_0}{\sqrt{2\pi\gamma t}p(1-p)} \exp\left[-\frac{\left(\frac{2}{\sqrt{\gamma}}\left(\ln\left[\frac{p}{1-p}\right] - \ln\left[\frac{p_0}{1-p_0}\right]\right) - \sqrt{\gamma}t/2\right)^2}{2t}\right]\mathrm{d}p \\
& + \frac{2(1-p_0)}{\sqrt{2\pi\gamma t}p(1-p)} \exp\left[-\frac{\left(\frac{2}{\sqrt{\gamma}}\left(\ln\left[\frac{p}{1-p}\right] - \ln\left[\frac{p_0}{1-p_0}\right]\right) + \sqrt{\gamma}t/2\right)^2}{2t}\right]\mathrm{d}p
\end{aligned}
$$



USING THE MEASUREMENT RESULTS

General feedback

State:

$$\mathrm{d}\rho_t = -i\,[H(y_{u<t}), \rho_t]\,\mathrm{d}t + \gamma\,\mathcal{D}[\mathcal{O}](\rho_t)\,\mathrm{d}t + \sqrt{\gamma}\,\mathcal{H}[\mathcal{O}](\rho_t)\,\mathrm{d}W_t$$

Signal:

$$\mathrm{d}y_t = \sqrt{\gamma}\,\mathrm{tr}\left[(\mathcal{O} + \mathcal{O}^\dagger)\,\rho_t\right]\,\mathrm{d}t + \mathrm{d}W_t$$

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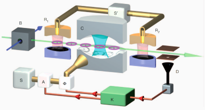
- Typically, one cannot get a closed form Master equation from this

State stabilisation:



Gleyzes et al. Nature **446**, 297-300 (2007)

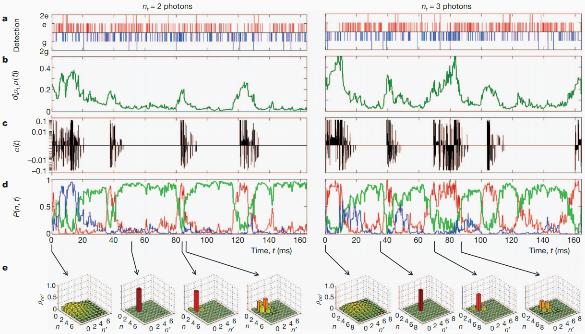


Figure 2 | Individual quantum feedback trajectories. Two feedback runs lasting 164 ms (2,000 loop iterations) stabilizing $|n_t = 2\rangle$ (left column) and $|n_t = 3\rangle$ (right column). The phase-shift per photon, $\phi_0 = 0.256\pi$, allows controller K to discriminate n values between 0 and 7. For $n_t = 2$, the Ramsey phase is $\phi_r = -0.44$ rad, corresponding to nearly equal e and g detection probabilities when $n = 2$. For $n_t = 3$, two Ramsey phases $\phi_{r,1} = -0.44$ rad and $\phi_{r,2} = -1.24$ rad are alternatively used, corresponding to equal e and g probabilities when $n = 2$ and $n = 3$, respectively. **a**, Sequences of qubit detection outcomes. The detection results are shown as blue downward bars for g and red upward bars for e. Two-atom detections in the same state appear as double-length bars. **b**, Estimated distance between the target and the actual state. **c**, Applied α-corrections (shown on a log scale as $\mathrm{sgn}(\alpha)\log|\alpha|$). **d**, Photon number probabilities estimated by K: $P(n = n_t)$ is in green, $P(n < n_t)$ in red, $P(n > n_t)$ in blue. **e**, Field density operators ρ in the Fock-state basis estimated by K at four different times marked by arrows.

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OTHER EXAMPLES

- **Faster purification**
  e.g. K. Jacobs, Phys. Rev. A 67, 030301(R) 2003
- **Faster measurement**
- **Continuous quantum error correction**
  e.g. C. Ahn, A. C. Doherty, and A. J. Landahl Phys. Rev. A 65, 042301 2002

Take the control proportional to the instantaneous signal:

$$H(t) \quad \text{``}=\text{''} \quad \frac{\mathrm{d}y_t}{\mathrm{d}t} \cdot \hat{C}$$

Take the control proportional to the instantaneous signal:

$$H(t) \quad \text{“}=\text{”} \quad \frac{\mathrm{d}y_t}{\mathrm{d}t} \cdot \hat{C}$$

The feedback can act only infinitesimally after the measurement:

$$\begin{aligned} \rho + \mathrm{d}\rho_{\mathrm{total}} &= U_{\mathrm{fb}} \left(\rho + \mathrm{d}\rho_{\mathrm{meas}}\right) U_{\mathrm{fb}}^{\dagger} \\ &= e^{-i\hat{C}\,\mathrm{d}y_t} \left(\rho + \mathrm{d}\rho_{\mathrm{meas}}\right) e^{i\hat{C}\,\mathrm{d}y_t} \end{aligned}$$

And we take the last line as a mathematical definition of Markovian feedback.

Physicist's cookbook

One can work with Itô processes by using only a few rules:

- $|\mathrm{d}W_t| \sim \sqrt{\mathrm{d}t}$
- $\mathbb{E}[\mathrm{d}W_t|\mathcal{F}_t] = 0$
- $\mathrm{d}W_t \, \mathrm{d}W_t = \mathrm{d}t$ $\simeq$ Itô's lemma

MARKOVIAN FEEDBACK

$$\begin{aligned}\rho + \mathrm{d}\rho_{\text{total}} &= e^{-i\hat{C}\,\mathrm{d}y_t}\left(\rho + \mathrm{d}\rho_{\text{meas}}\right)e^{i\hat{C}\,\mathrm{d}y_t} \\ &= \left(\mathbb{1} - i\hat{C}\mathrm{d}y_t - \frac{\hat{C}^2}{2}\underbrace{\mathrm{d}y_t^2}_{\mathrm{d}t}\right)\left(\rho + \mathrm{d}\rho_{\text{meas}}\right)\left(\mathbb{1} + i\hat{C}\mathrm{d}y_t - \frac{\hat{C}^2}{2}\underbrace{\mathrm{d}y_t^2}_{\mathrm{d}t}\right) \\ &= \cdots\end{aligned}$$

$$
\begin{aligned}
\rho + \mathrm{d}\rho_{\mathrm{total}} &= e^{-i\hat{C}\,\mathrm{d}y_t}\,(\rho + \mathrm{d}\rho_{\mathrm{meas}})\,e^{i\hat{C}\,\mathrm{d}y_t} \\
&= \left(\mathbb{1} - i\hat{C}\mathrm{d}y_t - \frac{\hat{C}^2}{2}\underbrace{\mathrm{d}y_t^2}_{\mathrm{d}t}\right)(\rho + \mathrm{d}\rho_{\mathrm{meas}})\left(\mathbb{1} + i\hat{C}\mathrm{d}y_t - \frac{\hat{C}^2}{2}\underbrace{\mathrm{d}y_t^2}_{\mathrm{d}t}\right) \\
&= \cdots
\end{aligned}
$$

Final form

After using the physicist version of Itô's formula one gets:

$$
\begin{aligned}
\mathrm{d}\rho_t =& \gamma\,\mathcal{D}[\mathcal{O}](\rho_t)\,\mathrm{d}t + \sqrt{\gamma}\,\mathcal{H}[\mathcal{O}](\rho_t)\,\mathrm{d}W_t \\
& - i[\hat{C}, \rho_t]\,\mathrm{d}W_t + \mathcal{D}[\hat{C}]\,\mathrm{d}t + -i\sqrt{\gamma}\left[\hat{C}, \mathcal{O}\rho_t + \rho_t\mathcal{O}^\dagger\right]\,\mathrm{d}t
\end{aligned}
$$

Markovian feedback master equation

$$
\begin{aligned}
\mathrm{d}\rho_t =& \gamma\,\mathcal{D}[\mathcal{O}](\rho_t)\,\mathrm{d}t \\
&+ \mathcal{D}[\hat{C}]\,\mathrm{d}t + -i\sqrt{\gamma}\left[\hat{C}, \mathcal{O}\rho_t + \rho_t\mathcal{O}^\dagger\right]\,\mathrm{d}t
\end{aligned}
$$

Markovian feedback master equation

$$\begin{aligned} \mathrm{d}\rho_t =& \gamma\,\mathcal{D}[\mathcal{O}](\rho_t)\,\mathrm{d}t \\ &+ \mathcal{D}[\hat{C}]\,\mathrm{d}t + -i\sqrt{\gamma}\left[\hat{C}, \mathcal{O}\rho_t + \rho_t\mathcal{O}^\dagger\right]\,\mathrm{d}t \end{aligned}$$

- Can be put into the Linblad form
- Has dissipation

SUMMARY

- One can construct a theory of continuous monitoring
- It is possible to feedback the measurement results
- A large number of applications
- In general one needs to work with the stochastic equations (no closed form equation for the average).