SPEEDING-UP CONTINUOUS MEASUREMENTS

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Inspired by a series of articles by Josh Combes and coauthors:

- arXiv:0712.3620
- arXiv:1105.0961
- arXiv:1410.8203

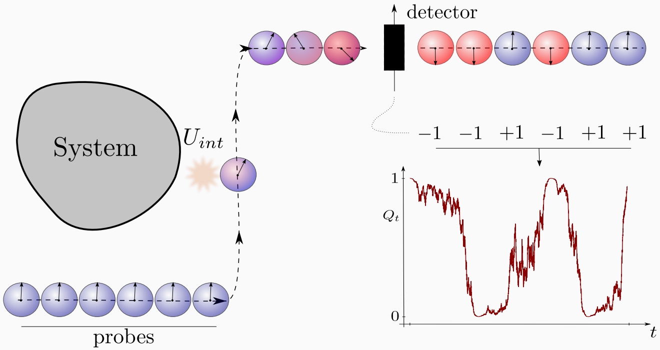
to which I made a modest contribution.

Side project corresponding to section 2.3 from my thesis (in french) and arXiv:1511.06555.

In a nutshell

With continuous measurements, measurements take time, so there is potential room for optimization via control.

How do you make a continuous measurement?



There are other ways of deriving the same results: weak coupling with infinite bosonic bath + unravelling, quantum noises, modified path integrals...

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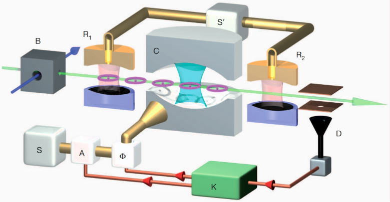
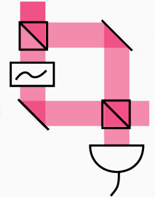
Ideal situations of application

- Discrete situations “a la Haroche”, with actual repeated interactions

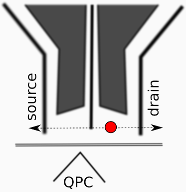


- “True” continuous measurement settings (homodyne detection in quantum optics)

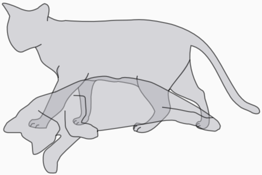


Other applications

- **Any** progressive measurement (e.g. quantum point contacts)



- Dynamical reduction models in quantum foundations



Continuous measurement equations

Continuous measurement of the operator $\mathcal{O}$ [Barchielli, Belavkin, Caves, Diósi, Milburn, Wiseman,...]:

$$\mathrm{d}\rho_t = -i[H, \rho_t]\,\mathrm{d}t + \mathcal{D}[\mathcal{O}](\rho_t)\,\mathrm{d}t + \mathcal{H}[\mathcal{O}](\rho_t)\,\mathrm{d}W_t \qquad (1)$$

with

- $\mathcal{D}[\mathcal{O}](\rho) = \mathcal{O}\rho\mathcal{O}^\dagger - \frac{1}{2}\{\mathcal{O}^\dagger\mathcal{O}, \rho\}$ usual Lindblad dissipator
- $\mathcal{H}[\mathcal{O}](\rho) = \mathcal{O}\rho + \rho\mathcal{O}^\dagger - \rho\,\mathrm{tr}[(\mathcal{O} + \mathcal{O}^\dagger)\rho]$ "stochastic innovation"
- W_t Wiener process i.e. for the lazy physicist "$\mathrm{d}W\mathrm{d}W = \mathrm{d}t$"

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Continuous measurement of a qubit

Take $\mathscr{H} = \mathbb{C}^2$, $\mathcal{O} \propto \sqrt{\gamma}\,\sigma_z$ and write $\rho_t = \begin{pmatrix} p_t & u_t \\ u_t^* & 1 - p_t \end{pmatrix}$. One gets in components:

- For the **probability**

$$dp_t = \sqrt{\gamma}\, p_t \,(1 - p_t)\, dW_t$$

- For the **phase**

$$du_t = -\frac{\gamma}{8} u_t \, dt + \frac{\sqrt{\gamma}}{2} (2p_t - 1)\, dW_t$$

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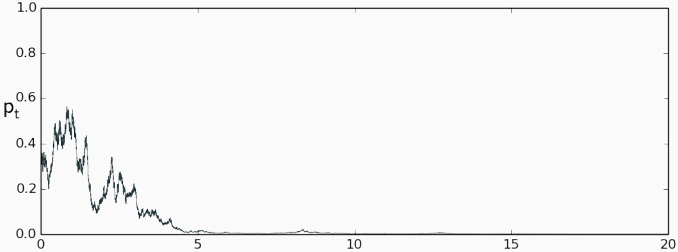
Probability

$dp_t = \sqrt{\gamma}\, p_t\,(1 - p_t)\, dW_t \rightarrow$ "collapse" towards the noise fixed points



Probability

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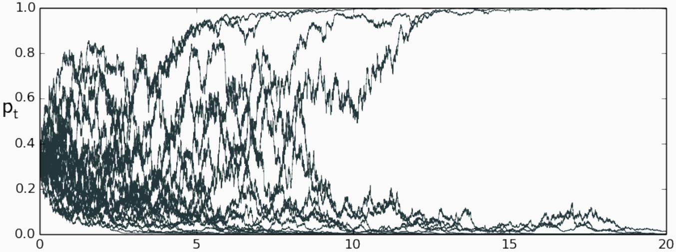


Phase

$$du_t = -\frac{\gamma}{8} u_t \, dt + \frac{\sqrt{\gamma}}{2}(2p_t - 1)\, dW_t$$

$$\Rightarrow \mathbb{E}[u_t] = e^{-\frac{\gamma t}{8}}$$

exponentially fast "dephasing"

This is generic

A continuous measurement always induces (in the measurement basis):

- **Progressive collapse** of the probabilities independently of the phases. Fully classical (same equation for Hidden Markov Models) = Bayesian updating
- **Progressive dephasing**, this is the essentially quantum part

With weak continuous measurements, new –previously meaningless– questions can be asked:

- How long does it take to reach some target entropy / purity / infidelity?
- Alternatively, what is the average entropy / purity / infidelity after some time T?
- Can this characteristic time be improved for **given** detector resources by toying with the system?

Initial formulation

$$d\rho_t = -i[H(t), \rho_t]\, dt + \mathcal{D}[\mathcal{O}](\rho_t)\, dt + \mathcal{H}[\mathcal{O}](\rho_t)\, dW_t$$

Play with the control Hamiltonian $H(t)$ –with $\mathcal{O}$ fixed– to minimize some information metric e.g.:

- $S(\rho_T) = -\mathrm{tr}(\rho_T \log \rho_T)$ for some final T $\rightarrow$ **global optimization**
- $\mathbb{E}[dS_L(\rho_t)|\mathcal{F}_t]$ with $S_L(\rho) = 1 - \mathrm{tr}(\rho^2)$ $\rightarrow$ **local optimization**

Other formulation

As we have no constraint on the norm of $H(t)$, changing H is equivalent to applying any unitary $U(t)$ on the system in real time so we have equivalently:

$$d\rho_t = \mathcal{D}[U(t)\mathcal{O}U^\dagger(t)](\rho_t)\,dt + \mathcal{H}[U(t)\mathcal{O}U^\dagger(t)](\rho_t)\,dW_t$$

Rotating the system is the same thing as rotating the detectors. Point of view often taken in the literature.

“Optimal” Purification

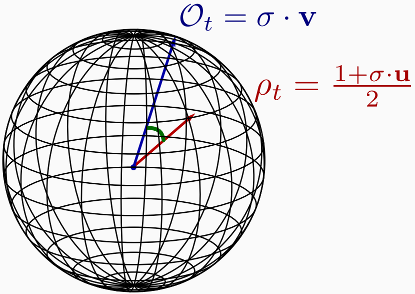
No constraint, everything is allowed. You can exploit the full quantumness of the state.

“Optimal” Measurement

The whole procedure needs to stay a measurement $\Rightarrow$ the only control operations allowed are permutation of vectors of the measurement basis. Fully classical if one looks only at the probabilities.

OPTIMAL PURIFICATION

Closed loop



The objective is to find **v** knowing **u** such that $\mathbb{E}[dS_L|\mathcal{F}_t]$ is minimal.

Closed loop

A straightforward application of Itô's formula gives:

$$\mathbb{E}[dS_L|\mathcal{F}_t] = -2\gamma\,(1 - \mathbf{u}\cdot\mathbf{v})(1 - (\mathbf{u}\cdot\mathbf{v})^2)\,dt \tag{2}$$

$\Rightarrow$ Optimality is reached for $\mathbf{u} \perp \mathbf{v}$ [Jacobs 2003]

Comments

In that case, $S_L(t) = e^{-4\gamma t}$ is **deterministic**!

Real time control needed!

Careful

It has been shown that this **locally optimal** scheme is also **globally optimal**. But some subtlety, with the optimal scheme:

- The average log-impurity at a given time decreases twice faster.
- One reaches a given log-impurity target twice slower on average.

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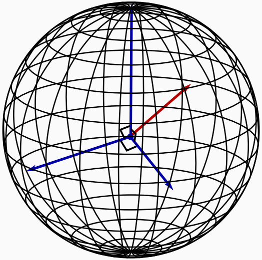
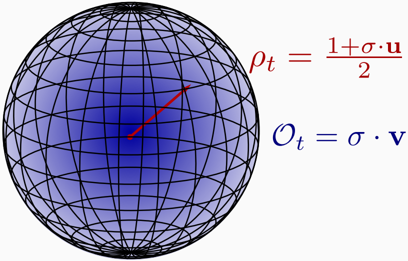
Open-loop purification

Try to be as orthogonal to the system as possible without knowing it. One can think of two possibilities:





Alternate between 3 orthogonal vectors or average over the whole Bloch sphere.

Closed-loop purification

In general, the locally optimal scheme is **not known**. In the case of the linear entropy, one needs to find U(t) such that:

$$\mathbb{E}[dS_L|\mathcal{F}_t] = \mathrm{tr}\left[2\rho_t\mathcal{D}[U(t)\mathcal{O}U^\dagger(t)](\rho_t) + \mathcal{H}[U(t)\mathcal{O}U^\dagger(t)](\rho_t)^2\right]\, dt$$

is extremal.

Closed-loop purification

The case where $\mathcal{O}$ has linearly spaced eigenvalues is the only one that has been studied. It is known that:

- The speed-up is at least $\propto n^2$ where $n = \dim \mathscr{H}$
- Deterministic information extraction holds.
- Complementarity ($\simeq$ orthogonality) is not enough and can lead to a speed-up factor as low as 2.
- It seems that the part of the speed-up scaling up with dimension is of classical origin as it is the same for optimal measurement protocols.

Open-loop purification

One can also choose the unitaries randomly (say uniformly with the Haar measure):

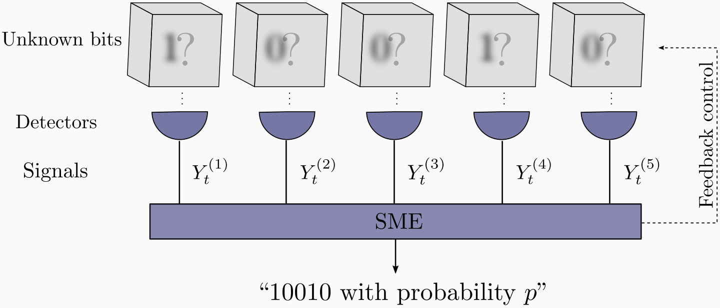
$$\mathbb{E}[dS_L|\mathcal{F}_t] = \int_{U\in\mathcal{U}(n)} dU \, \mathrm{tr}\left[2\rho_t\mathcal{D}[U\mathcal{O}U](\rho_t) + \mathcal{H}[U\mathcal{O}U](\rho_t)^2\right] \, dt$$

Seems to be the best open-loop one can do. (Proof?)

OPTIMAL MEASUREMENT

Setup

A qubit is too trivial → consider a qubit register:



Quantum mechanical writing

Consider N qubits, i.e. $\mathscr{H} = \left(\mathbb{C}^2\right)^{\otimes N}$. Measure simultaneously $\sigma_z^j = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \sigma_z \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$:

$$d\rho_t = \sum_{j=1}^{N} \mathcal{D}\left[\sigma_z^j\right](\rho_t)\, dt + \mathcal{H}\left[\sigma_z^j\right](\rho_t)\, dW_t^j,$$

with $dW_t^i\, dW_t^j = \delta_{i,j}\, dt$, i.e. **independent** measurements. Start for simplicity with no information, i.e. $\rho_0 = \frac{\mathbb{1}}{2^N}$.

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Classical (equivalent!) picture

You have N classical bits with a well defined but unknown value. You start measuring them progressively and independently (take blurry pictures). Write:

$$p_t^j = \mathbb{P}[\text{bit } j = 1 | \text{all measurements before } t]$$

Then:

$$dp_t^j = 4p_t^j(1 - p_t^j)\, dW_t^j$$

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QUBIT REGISTER

Idea

Why is some speed-up expected?

Because the no-control measurement procedure is bad at distinguishing the most probable configuration from the second most probable one.

Indeed because of independence, these two states look like this:

$$\begin{matrix} 101000\boxed{0}0101110 \\ 101000\boxed{1}0101110 \end{matrix}$$

The Hamming distance between the two is only 1.

Idea

Only 1 out of the N detector is actually useful to discriminate between two states with Hamming distance 1.

A good scheme should be able to use them all at once and provide a speed-up $\propto$ N.

Locally optimal scheme

Reorder the pointer basis in real time, e.g.:

$$
\begin{array}{ccc}
\text{most probable} & |1011101\rangle & \rightarrow & |1111111\rangle \\
\\
2^{\text{nd}} & |0011101\rangle & \rightarrow & |0000000\rangle \\
3^{\text{rd}} & |1001101\rangle & \rightarrow & |1000000\rangle \\
4^{\text{th}} & |1011111\rangle & \rightarrow & |0100000\rangle \\
\dots & \dots & \rightarrow & \dots \\
\text{last} & |0100010\rangle & \rightarrow & |1111110\rangle
\end{array}
$$

Exact speed-up not known but $\geq \frac{N}{4}$.

Open-loop scheme

Do fast random permutations of the pointer basis:

open loop purification | **open loop measurement**

$$\int_{U\in\mathcal{U}(n)} dU \quad \longrightarrow \quad \sum_{\sigma\in\mathfrak{S}(2^N)}$$

Similar speed-up $\propto N$ (actually $= \frac{N}{2}$).

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Problems

The previous schemes are **impossible to implement** –even only numerically– on registers of size $N \geqslant 10$.

- Need 2^N instead of N variables to store the probability.
- For open loop, need to sample $(2^N)!$ permutations.

Is it possible to solve these problems while keeping a linear speed-up?

New algorithm

One can try to implement a few frugality constraints:

- Use a smaller number of basis!
- Have the permutation map between the basis be simple.
- Ideally, have all states far away from each other on average with respect to the Hamming distance.

Last constraint is too strong seems impossible to implement (Singleton bound?)

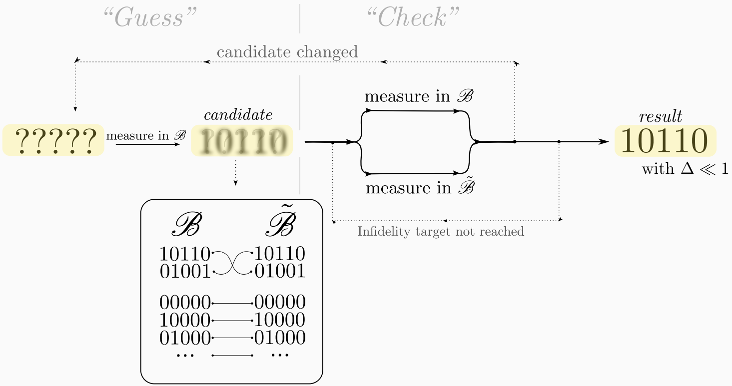
Protocol

The idea is to run the standard measurement procedure for a while until a reasonable candidate is reached and construct a new basis where it is bit flipped:

$$
\begin{array}{cccc}
\text{candidate} & |1011101\rangle & \rightarrow & |0100010\rangle \\
\text{bitwise flipped} & |0100010\rangle & \rightarrow & |1011101\rangle \\
 & & & \\
\text{rest} & |1000000\rangle & \rightarrow & |1000000\rangle \\
 & |0100000\rangle & \rightarrow & |0100000\rangle \\
\ldots & \ldots & \rightarrow & \ldots
\end{array}
$$

Then measure in the original basis and the new alternatively.

Protocol

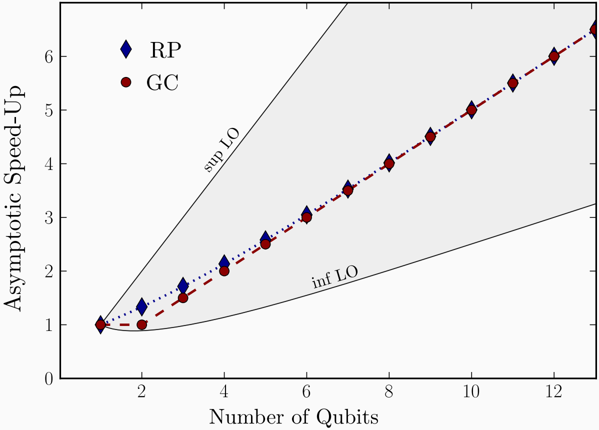

"Guess"
"Check"
candidate changed
?????
measure in $\mathscr{B}$
candidate
10110
measure in $\mathscr{B}$
measure in $\tilde{\mathscr{B}}$
result
10110
with $\Delta \ll 1$
Infidelity target not reached
$\mathscr{B}$
$\tilde{\mathscr{B}}$
10110
01001
10110
01001
00000
10000
01000
...
00000
10000
01000
...

Things you need to prove

- Prove that it is possible to reconstruct all the probabilities with the 2N marginals and a small number of operations.
- Prove that the “guess” phase has no impact on the asymptotic speed-up.
- Compute the asymptotic speed-up in the “check” phase

Actually OK and asymptotic speed-up $= \frac{N}{2}$.

Asymptotic speed-up



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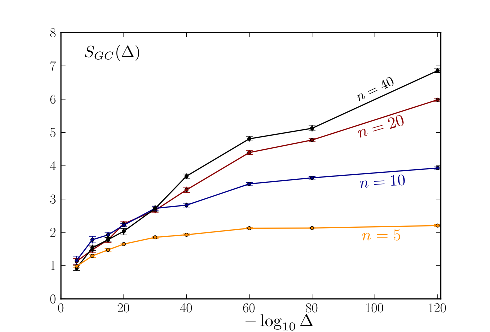
QUBIT REGISTER

Limitations

Practical speed-up much worth than asymptotic speed-up:



A few open questions

- What about global optimality?
- What about short time behavior?
- What about more generic systems?
- What is quantum and what is classical in optimal purification?