Continuous Tensor Network States for Quantum Fields

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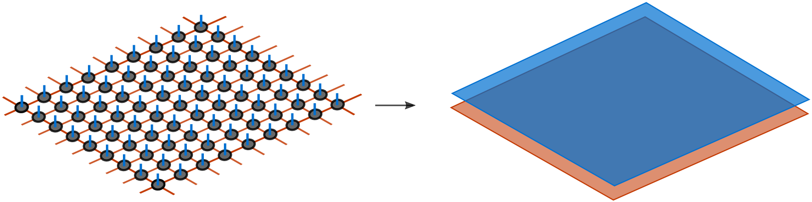
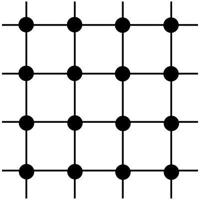


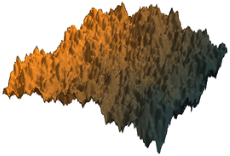
MPQ Theory division group workshop
Nordlingen, Germany
October 19th, 2018

Discrete vs continuum theories



$$Z(\beta) = \sum_s e^{-\beta E(s)}$$

with $E(s) = \sum_{k,\ell} S_k S_\ell$



$$Z(\beta) = \int \mathcal{D}\phi e^{-\int \mathcal{L}(\phi)}$$

with $\mathcal{L}(\phi) = \frac{(\nabla\phi)^2}{2} + \frac{m^2\phi^2}{2} + \lambda\phi^4$

Lots of "Continuous tensor network" concepts

Tensor networks for quantum states $|\psi\rangle$

- 
  MPS $\rightarrow$ cMPS
  [Verstraete & Cirac 2010]

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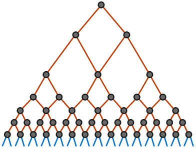
- 

  MERA $\rightarrow$ cMERA

  [Haegeman et al. 2013]

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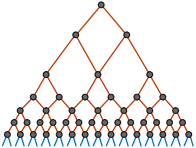
- 
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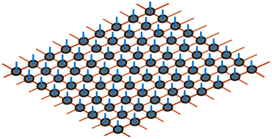
- 
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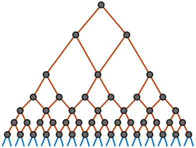
- 
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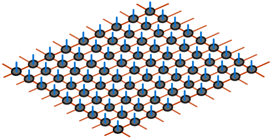
- 
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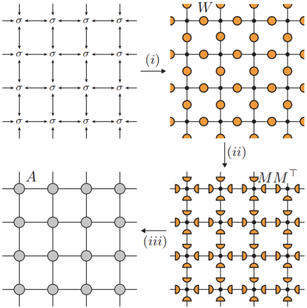
Tensor networks for partition functions $Z(\beta)$

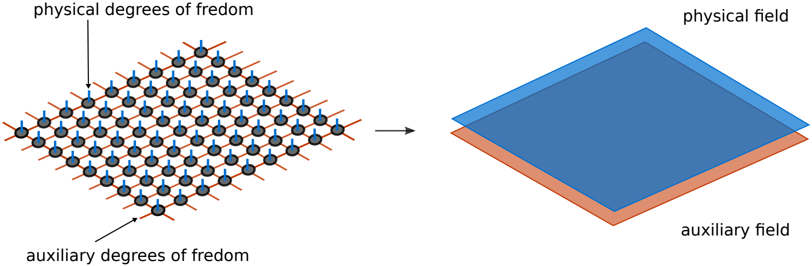
- StatMech in d
- Euclidean quantum in $d+1$



[Franco-Rubio et al. 2018]

Objective

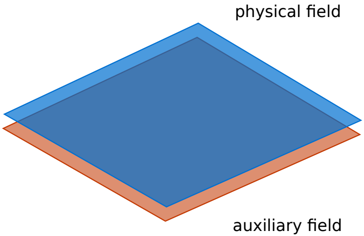
Define a **continuous** tensor network **state** for $d \geqslant 2$



Objective

Why?

- **Trickiness** of $d \geqslant 2$
- **Computations**: the continuum brings new methods (perturbative expansions, saddle point approximations, differential equations)
- **QFT**: apply directly to QFT, without discretization
- **Symmetries**: Implement Euclidean / Translation invariance exactly
- **Holography**: (?) Construct better toy models



Matrix product states

$$|\psi\rangle = \sum_{i_1, i_2, \cdots, i_n = \pm 1} c_{i_1, i_2, \cdots, i_n} \, |i_1, \cdots, i_n\rangle$$

Matrix Product States (MPS)

$$|A, L, R\rangle = \sum_{i_1, i_2, \cdots, i_n} \langle \mathrm{L}| A_{i_1}(1) A_{i_2}(2) \cdots A_{i_n}(n) |\mathrm{R}\rangle \, |i_1, \cdots, i_n\rangle$$

- A_i, $i = \pm 1$ are $D \times D$ complex matrices
- A is a $2 \times D \times D$ tensor $[A_i]_{k,l}$
- $|L\rangle$ and $|R\rangle$ are D-vectors.

♢ $n \times 2 \times D^2$ parameters instead of 2^n

♢ D is the **bond dimension** and encodes the size of the variational class

Graphical notation

$$|A,L,R\rangle = \sum_{i_1,i_2,\cdots,i_n} \langle \mathrm{L}|A_{i_1}(1)A_{i_2}(2)\cdots A_{i_n}(n)|\mathrm{R}\rangle \; |i_1,\cdots,i_n\rangle$$

Notation: $[A_i]_{\mathbf{k},\mathbf{l}} =$  and k —— $l = \sum \delta_{k,l}$ gives:

$|A,L,R\rangle =$ 

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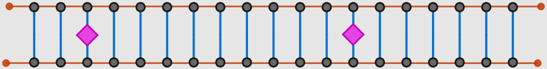
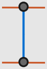
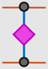
Example: computation of correlations

$\langle A|\mathcal{O}(i_k)\mathcal{O}(i_\ell)|A\rangle =$ 

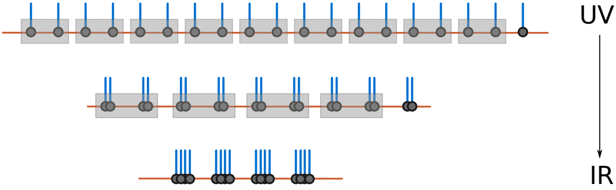
can be done by iteration 2 maps:

$\Phi =$  and $\Phi_{\mathcal{O}} =$ 

Contraction for a $d = 1$ system $\sim$ open-system dynamics in $d = 0$.

Continuous Matrix Product States (cMPS)

Taking the continuum limit of a MPS



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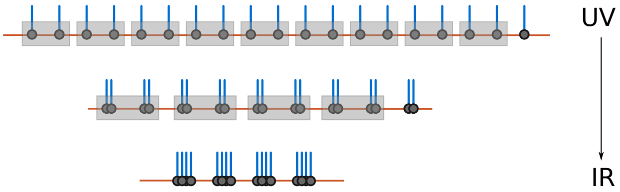

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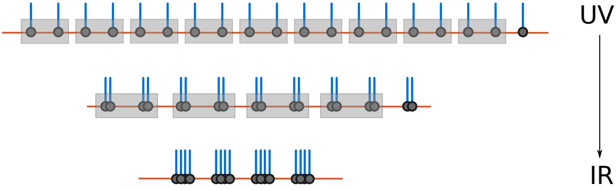

- the bond dimension D stays fixed
- the local physical dimension explodes $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \longrightarrow \mathscr{F}(L^2([x, x+dx]))$.

  $\Longrightarrow$ **Spins** become **fields** – ($\simeq$ central limit theorem)

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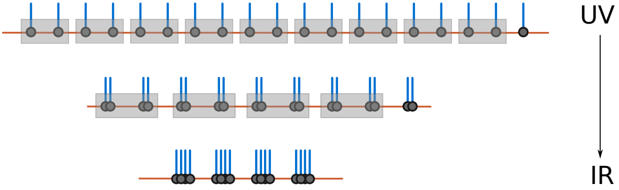

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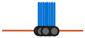
  $\Longrightarrow$ **Spins** become **fields** – ($\simeq$ central limit theorem)
- A cMPS is a quantum field state parameterized by finite dimensional matrices

Continuous Matrix Product States

Type of ansatz

- Matrices $A_{i_k}(x) =$  where the index i_k corresponds to

$$\varepsilon^{-i_k/2} a^{\dagger i_k}(x)|0\rangle = \psi^{\dagger i_k}(x)|0\rangle$$

  in **physical space**.

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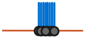
- Matrices $A_{i_k}(x) =$  where the index i_k corresponds to

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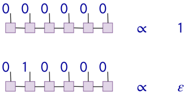
Informal cMPS definition

$$A_0 = \mathbb{1} + \varepsilon Q$$
$$A_1 = \varepsilon R$$
$$A_2 = \frac{(\varepsilon R)^2}{\sqrt{2}}$$
$$\ldots$$
$$A_n = \frac{(\varepsilon R)^n}{\sqrt{n}}$$
$$\ldots$$

so we go from ∞ to 2 matrices

Fixed by:

- Finite particle number



- Consistency

Continuous Matrix Product States

Definition

$$|Q, R, \omega\rangle = \langle\omega_L|\mathcal{P}\exp\left\{\int_0^L dx\ Q\otimes\mathbb{1} + R\otimes\psi^\dagger(x)\right\}|\omega_R\rangle\,|0\rangle$$

- Q, R are $D\times D$ matrices,
- $|\omega_L\rangle$ and $|\omega_R\rangle$ are boundary vectors $\in\mathbb{C}^D$,
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Idea:

$$\begin{aligned} A(x) &\simeq A_0 \mathbb{1} + A_1 \psi^\dagger(x) \\ &\simeq \mathbb{1} \otimes \mathbb{1} + \varepsilon Q \otimes \mathbb{1} + \varepsilon R \otimes \psi^\dagger(x) \\ &\simeq \exp \left[\varepsilon \left(Q \otimes \mathbb{1} + R \otimes \psi^\dagger(x) \right) \right] \end{aligned}$$

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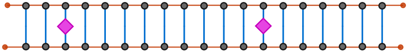
Computations

Thermodynamic limit

$$\langle\mathcal{O}(x)\mathcal{O}(y)\rangle = \mathrm{tr}\left(\Phi_{\mathcal{O}} \cdot e^{-(y-x)T}\Phi_{\mathcal{O}} \cdot \rho_{\mathrm{stat}}\right)$$

with $T = Q\otimes\mathbb{1} + \mathbb{1}\otimes\bar{Q} + R\otimes\bar{R}$



Extending continuous matrix product states

$$|Q, R, \omega\rangle = \langle \omega_L | \mathcal{P} \exp \left\{ \int_0^L dx \; Q \otimes \mathbb{1} + R \otimes \psi^\dagger(x) \right\} |\omega_R\rangle \; |0\rangle$$

Besides that, ***it is possible to extend this formalism to 2-dimensional continuum systems*** *using the formalism of PEPS [8]. In that case, the auxiliary bond dimension has to be interpreted as representing an auxiliary field, and* ***the judicious choice of tensors Q and R allows to develop a consistent formalism for describing $2+1$ dimensional field theories*** *[10].*

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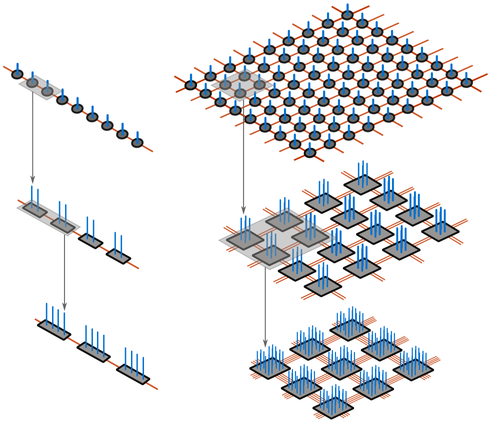
[9] E. H. Lieb and W. Liniger, *Phys. Rev.* **130**, 1605 (1963)
[10] F. Verstraete and J. I. Cirac, in preparation.
[11] G.E. Astraharchik and S. Giorgini, *Phys. Rev. A* **68**, 031602 (2003)

Continuous Tensor Networks: blocking



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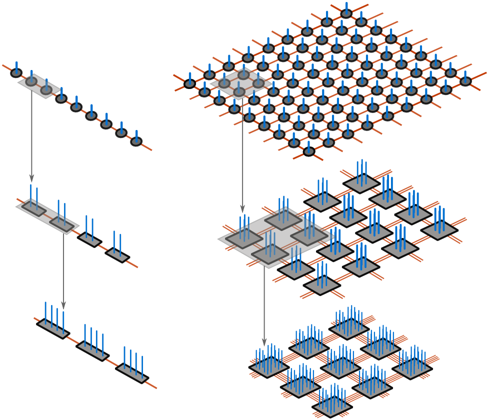
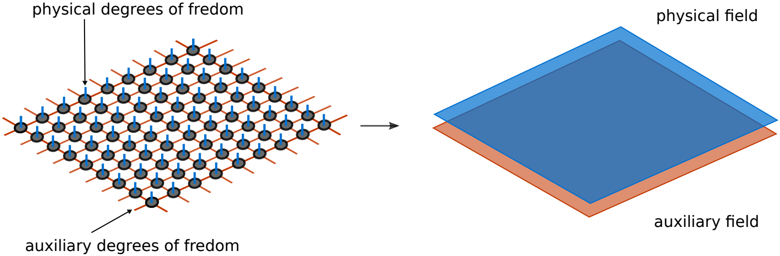


Upon blocking:

- ♣ The **physical** Hilbert space dimension *d* increases (idem cMPS $\Longrightarrow$ physical field)
- ♣ The **bond** dimension *D* increases too

Foreshadowing



$$|V,\alpha\rangle = \int \mathcal{D}\phi\ \exp\left\{-\int_\Omega \mathrm{d}^d x\ \frac{1}{2}\left[\nabla\phi(x)\right]^2 + V[\phi(x)] - \alpha[\phi(x)]\,\psi^\dagger(x)\right\}|0\rangle$$

Choice of trivial tensor

For **MPS**, not much choice:

 $= \mathbb{1} \otimes \mathbb{1}\,|0\rangle + \varepsilon Q \otimes \mathbb{1}\,|0\rangle \;+\; \varepsilon R \otimes \psi^\dagger\,|0\rangle$

$=$ $+\;\; \varepsilon \cdots$

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For **TNS** in $d \geqslant 2$, many options:

1. Take a δ between all legs $\sim$ GHZ state $T^{(0)} =$ 
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3. Take the sum of pairs of identities in both directions
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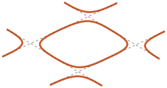

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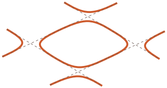


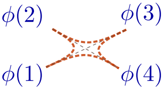
We will consider a softer modification of the first version:

$T^{(0)} \sim$ 

Ansatz

1 – Take a "Trivial" tensor:

$$T^{(0)}_{\phi(1),\phi(2),\phi(3),\phi(4)} =$$



$$\sim \exp\Big\{ -[\phi(1)-\phi(2)]^2 - [\phi(2)-\phi(3)]^2 - [\phi(3)-\phi(4)]^2 - [\phi(4)-\phi(1)]^2 \Big\}$$

The indices ϕ are in $\mathbb{R}$ (and **not** $1, \cdots, D$)

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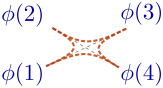

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2 – And add a "correction":

$$\exp\left\{-\varepsilon^2 V\left[\phi(1),\cdots,\phi(4)\right] + \varepsilon^2 \alpha\left[\phi(1),\cdots,\phi(4)\right]\psi^\dagger(x)\right\}$$

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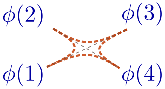

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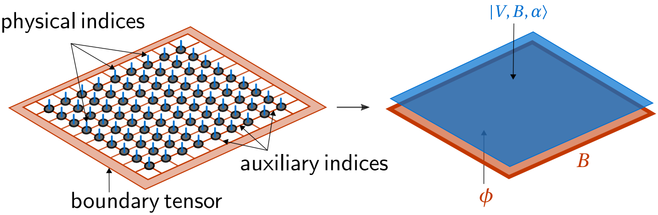
3 – Realize tensor contraction = functional integral and trivial tensor gives free field measure.

Functional integral definition

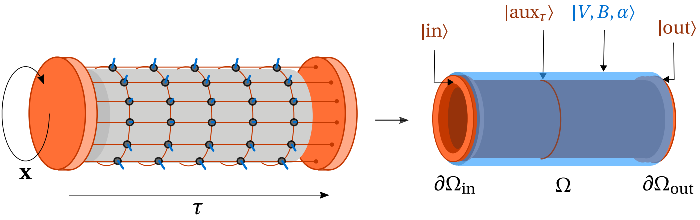


Continuous tensor network state (cTNS)

A cTNS is a **state** parameterized by 2 functions V, α and a functional B:

$$|V, B, \alpha\rangle = \int \mathcal{D}\phi \, B(\phi|_{\partial\Omega}) \exp\left\{ -\int_{\Omega} \mathrm{d}^d x \, \frac{1}{2} \sum_{k=1}^{D} [\nabla \phi_k(x)]^2 + V[\phi(x)] - \alpha[\phi(x)] \, \psi^\dagger(x) \right\} |0\rangle$$

Operator definition



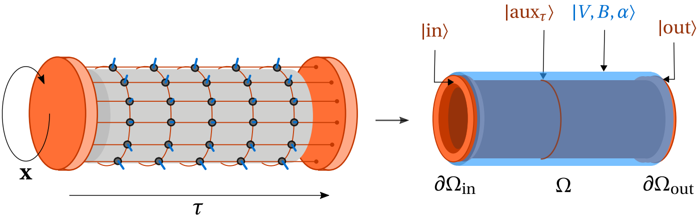
Continous tensor network state (cTNS)

$$|V,B,\alpha\rangle = \mathrm{tr}\left[\hat{B}\,\mathcal{T}\exp\left(-\int_0^T d\tau \int_S dx\ \frac{\hat{\pi}_k(x)\hat{\pi}_k(x)}{2} + \frac{\nabla\hat{\phi}_k(x)\nabla\hat{\phi}_k(x)}{2} + V[\hat{\phi}(x)] - \alpha[\hat{\phi}(x)]\,\psi^\dagger(\tau,x)\right)\right]|0\rangle$$

where:

- $\hat{\phi}_k(x)$ and $\hat{\pi}_k(x)$ are k independent canonically conjugated pairs of (auxiliary) field *operators*: $[\hat{\phi}_k(x),\hat{\phi}_l(y)] = 0$, $[\hat{\pi}(x)_k,\hat{\pi}_l(y)] = 0$, and $[\hat{\phi}_k(x),\hat{\pi}_l(y)] = i\delta_{k,l}\,\delta(x-y)$ acting on a space of $d-1$ dimensions.

Operator definition



Continous tensor network state (cTNS)

$$|V, B, \alpha\rangle = \text{tr}\left[\hat{B}\,\mathcal{T}\exp\left(-\int_0^T d\tau \int_S dx\ \frac{\hat{\pi}_k(x)\hat{\pi}_k(x)}{2} + \frac{\nabla\hat{\phi}_k(x)\nabla\hat{\phi}_k(x)}{2} + V[\hat{\phi}(x)] - \alpha[\hat{\phi}(x)]\,\psi^\dagger(\tau, x)\right)\right]|0\rangle$$

where:

- Morally: $Q \sim \frac{\hat{\pi}_k(x)\hat{\pi}_k(x)}{2} + \frac{\nabla\hat{\phi}_k(x)\nabla\hat{\phi}_k(x)}{2} + V[\hat{\phi}(x)]$ and $R \sim \alpha[\hat{\phi}(x)]$

Wave-function definition

A generic state $|\Psi\rangle$ in Fock space can be written:

$$|\Psi\rangle = \sum_{n=0}^{+\infty} \int_{\Omega^n} \frac{\varphi_n(x_1, \cdots, x_n)}{n!} \, \psi^\dagger(x_1) \cdots \psi^\dagger(x_n) \, |0\rangle$$

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Physical wave-function   correlation function of the auxiliary field:

$$\varphi(x_1, x_2, \cdots, x_n) = \langle \, \alpha[\phi(x_1)] \, \alpha[\phi(x_2)] \cdots \alpha[\phi(x_n)] \, \rangle$$

Functional integral representation

- $\langle \clubsuit \rangle = \int \mathcal{D}\phi \, e^{-S(\phi)} \, \clubsuit$

♡ Extension of Moore-Read

Operator representation

- $\langle \clubsuit \rangle = \mathrm{tr}\left[\hat{B} \clubsuit\right]$
- $\alpha[\phi(x)] = \alpha[\hat{\phi}(x)]$ in (imaginary time) interaction representation

Expressivity and stability

How big are cTNS?

Stability

The sum of two cTNS of bond field dimension D_1 and D_2 is a cTNS with bond field dimension $D \leqslant D_1 + D_2 + 1$:

$$|V_1, \alpha_1\rangle + |V_2, \alpha_2\rangle = |W, \beta\rangle$$

Expressiveness

All states in the Fock space can be approximated by cTNS:

- A field coherent state is a cTNS with $D = 0$
- Stability allows to get all sums of field coherent states

Note: expressiveness can also be obtained with $D = 1$. Flexibility in D makes the expressivity higher for V and α fixed degree.

Computations

$$|V, B, \alpha\rangle = \int \mathcal{D}\phi \, B(\phi|_{\partial\Omega}) \exp\left\{-\int_\Omega \mathrm{d}^d x \, \frac{1}{2} \sum_{k=1}^{D} [\nabla \phi_k(x)]^2 + V[\phi(x)] - \alpha[\phi(x)] \, \psi^\dagger(x)\right\} |0\rangle$$

Gaussian cTNS

If:

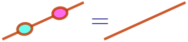
$$V(\phi) = V^{(0)} + V_k^{(1)} \phi_k + V_{k\ell}^{(2)} \phi_k \phi_\ell$$
$$\alpha(\phi) = \alpha^{(0)} + \alpha_k^{(1)} \phi_k$$

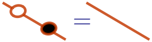
then $|V, \alpha, B\rangle$ is a Gaussian state

Redundancies

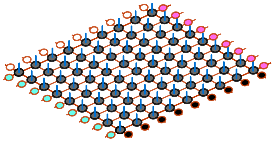
Discrete redundancy

Different elementary tensors are **equivalent**, they give the same state:

when  and



up to **boundary** terms:



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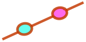
when  $=$  and  $=$ 

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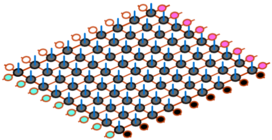
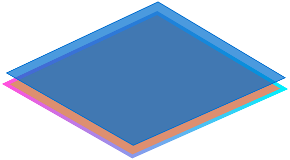


Continuum redundancy

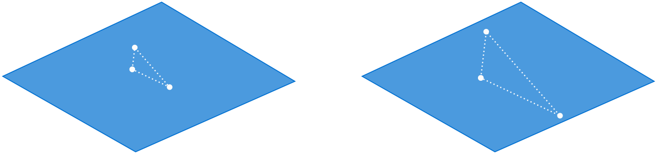
$$V(\phi) \to V(\phi) + \nabla \cdot \mathscr{F}[x, \phi(x)]$$

Just Stokes' theorem. If Ω has a boundary $\partial\Omega$:

$$\mathcal{D}[\phi] \exp\left\{ \oint_{\partial\Omega} \mathrm{d}^{d-1}x\, \mathscr{F}[x, \phi(x)] \cdot \mathbf{n}(x) \right\}$$



Renormalization / scaling



$$C(x_1, \cdots, x_n) = \langle T(1) | \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) | T(1) \rangle,$$

the objective is to find a tensor $T(\lambda)$ of new parameters such that:

$$C(\lambda x_1, \cdots, \lambda x_n) \propto \langle T(\lambda) | \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) | T(\lambda) \rangle.$$

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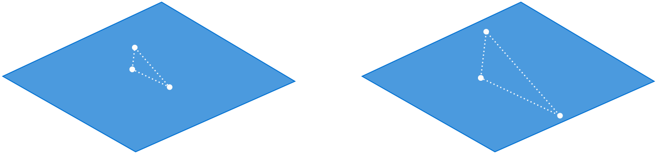

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Doable exactly:

$$V \to \lambda^d V \circ \lambda^{\frac{2-d}{2}} \quad \text{and} \quad \alpha \to \lambda^{\frac{d}{2}} \alpha \circ \lambda^{\frac{2-d}{2}}$$

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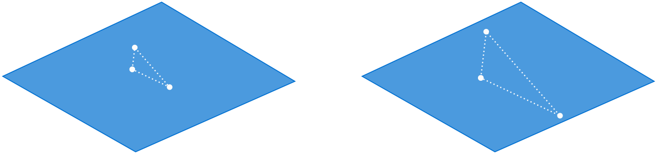

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- $d = 2$, All powers of the field in V and α yield relevant couplings
- $d = 3$, The powers $p = 1, 2, 3, 4, 5$ of the field in V yield relevant $\Delta > 0$ couplings. $p = 6$ is marginal in V. For α, $p = 1, 2$ are relevant and $p = 3$ is marginal. All other p are irrelevant.

Getting back cMPS

One can get back cMPS with finite bond dimension by:

1. **Compactification** Take $d-1$ dimensions out of d to be very small

$$|V, B, \alpha\rangle \simeq \text{tr}\left[\hat{B}\,\mathcal{T}\exp\left(-\int_0^T \mathrm{d}\tau \sum_{k=1}^{D} \frac{\hat{P}_k^2}{2} + V[\hat{X}] - \alpha[\hat{X}]\,\psi^\dagger(\tau)\right)\right]|0\rangle$$

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2. **Quantization** Take V with D deep minima to force the auxiliary field to take only D possibilities

Generalization

For a general Riemanian manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, define:

$$|V, B, \alpha\rangle = \int \mathcal{D}\phi \, B(\phi|_{\partial\mathcal{M}}) \exp\left\{ -\int_{\mathcal{M}} \mathrm{d}^d x \sqrt{g} \left(\frac{g^{\mu\nu}\partial_\mu \phi_k \partial_\nu \phi_k}{2} \right.\right.$$
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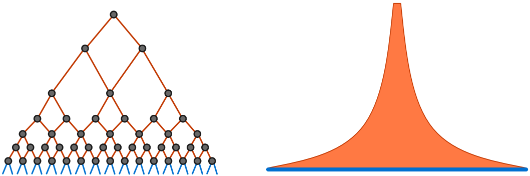
Example: $\alpha[x, \phi, \nabla\phi]$ localized on the boundary and hyberbolic metrix g:



$\rightarrow$ **cMERA** in $d-1$ dimensions

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- **Non-trivial Non-Gaussian states?**

Summary

$$|V,B,\alpha\rangle = \int \mathcal{D}\phi \; \exp\left\{ -\int_{\Omega} \mathrm{d}^d x \; \frac{1}{2}\left[\nabla\phi(x)\right]^2 + V[\phi(x)] - \alpha[\phi(x)]\,\psi^\dagger(x) \right\} |0\rangle$$

Continuous tensor network states are natural continuum limits of tensor network states and natural higher d extensions of continuous matrix product states.

1. Obtained from discrete tensor networks
2. Can be made Euclidean invariant
3. Have functional and operator representations
4. Have a geometrical equivalent of the discrete gauge redundancies
5. Have an exact and explicit "renormalization" flow