Continuous Tensor Network States of Quantum Fields

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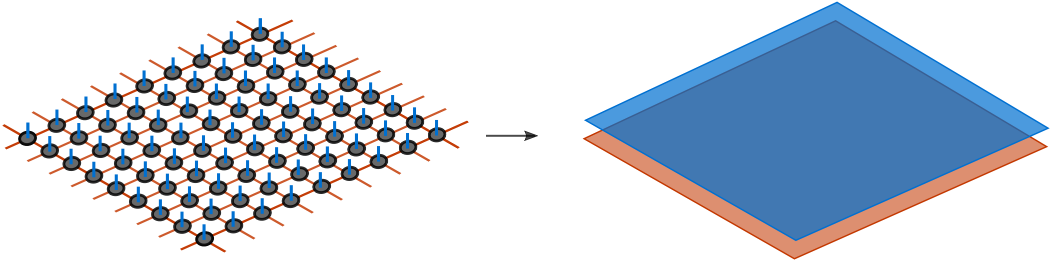

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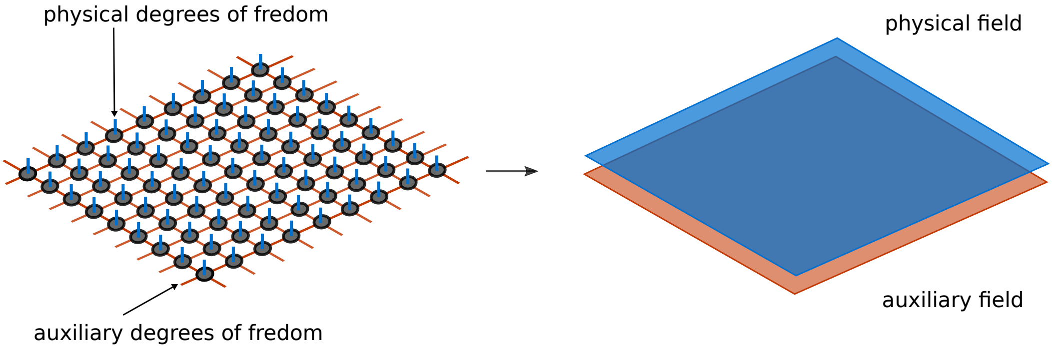
Alexander von Humboldt
Stiftung/Foundation

MPQ

Objective

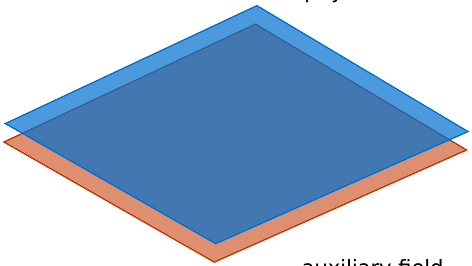


$$|V,\alpha\rangle = \int \mathcal{D}\phi \ \exp\left\{-\int_\Omega \mathrm{d}^d x \ \frac{1}{2}\sum_{k=1}^{D} [\nabla\phi_k(x)]^2 + V[\phi(x)] - \alpha[\phi(x)]\ \hat{\psi}^\dagger(x)\right\} |0\rangle$$

Objective

Why?

- **Trickiness of** $d \geqslant 2$
- **Computations**: the continuum brings new methods (perturbative expansions, saddle point approximations, differential equations)
- **QFT**: apply directly to QFT, without discretization
- **Symmetries**: Implement Euclidean / Translation invariance exactly
- **Holography**: (?) Construct better toy models



Problem

Many-body states are complicated.

$$|\psi\rangle = \sum_{i_1, i_2, \cdots, i_n} c_{i_1, i_2, \cdots, i_n} |i_1, \cdots, i_n\rangle$$

2^n parameters $c_{i_1, i_2, \cdots, i_n}$.

Typical many-body Hamiltonians are simple.

$$H = \sum_{k=1}^{n} h_k$$

$\sim$ const $\times$ n parameters.

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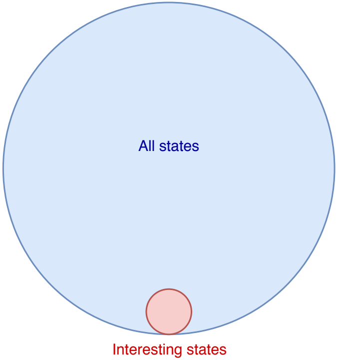

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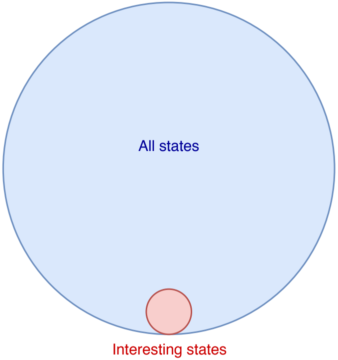


Variational optimization

To find the ground state:

$$|\text{ground}\rangle = \min_{|\psi\rangle \in \mathscr{S}} \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

Can we find a subspace $\mathscr{S}$ s. t.:

- $|\mathscr{S}| \propto n^k \ll e^n$
- $\mathscr{S}$ approximates well interesting states
- *bonus* $\langle\psi|\mathcal{O}(x)|\psi\rangle$ is computable

An idea popular in many fields

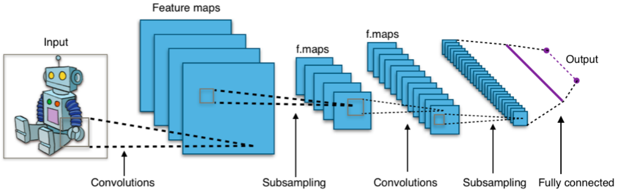
- **Mean field** approximation (of which TNS are an extension)

$$\psi(x_1, x_2, \cdots, x_n) = \psi_1(x_1)\,\psi_2(x_2)\cdots\psi_n(x_n)$$

- Special variational wave functions in **Quantum chemistry** (whole industry of ansatz)
- **Moore-Read wavefunctions** in the study of the quantum Hall effect

$$\psi(x_1, x_2, \cdots, x_n) = \left\langle \hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n)\right\rangle_{\text{CFT}}$$

- Fully connected and convolutional **neural networks** used in machine learning



Matrix product states

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Matrix Product States (MPS)

$$|A, L, R\rangle = \sum_{i_1, i_2, \cdots, i_n} \langle \mathrm{L}|A_{i_1}(1) A_{i_2}(2) \cdots A_{i_n}(n)|\mathrm{R}\rangle \, |i_1, \cdots, i_n\rangle$$

- A_i are $D \times D$ complex matrices
- A is a $2 \times D \times D$ tensor $[A_i]_{k,l}$
- $|L\rangle$ and $|R\rangle$ are D-vectors.

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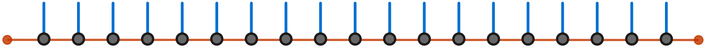
Remark: actually equivalent with the density matrix renormalization group (DMRG)

- ♢ $n \times 2 \times D^2$ parameters instead of 2^n
- ♢ D is the **bond dimension** and encodes the size of the variational class

Graphical notation

$$|A, L, R\rangle = \sum_{i_1, i_2, \cdots, i_n} \langle \mathrm{L}| A_{i_1}(1) A_{i_2}(2) \cdots A_{i_n}(n) |\mathrm{R}\rangle \; |i_1, \cdots, i_n\rangle$$

Notation: $[A_i]_{\mathbf{k},\mathbf{l}} =$  and k —— $l = \sum \delta_{k,l}$ gives:

$|A, L, R\rangle =$ 

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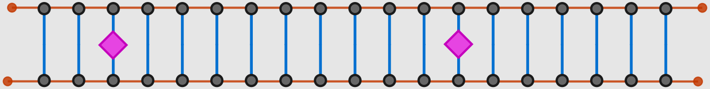
Example: computation of correlations

$\langle A|\mathcal{O}(i_k)\mathcal{O}(i_\ell)|A\rangle =$ 

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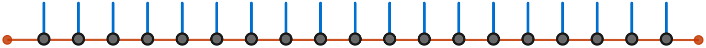
$|A, L, R\rangle =$ 

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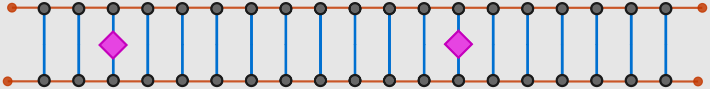
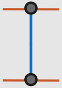
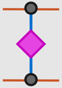
$\langle A|\mathcal{O}(i_k)\mathcal{O}(i_\ell)|A\rangle =$ 

can be done efficiently by iterating 2 maps:

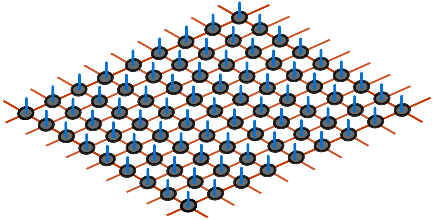
$\Phi =$  and $\Phi_{\mathcal{O}} =$ 

The contraction for a $d = 1$ system, can be seen as an open-system dynamics in $d = 0$.

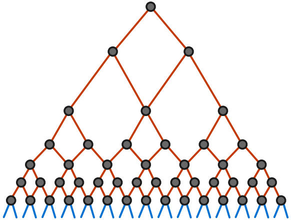
Generalizations: different tensor networks

Matrix Product States (MPS)

Projected Entangled Pair States (PEPS)



Multi-scale Entanglement Renormalization Ansatz (MERA)



Some facts

A list of theorems [very colloquially]:

- **Expressiveness** [trivial] Tensor Network States cover $\mathscr{H}$ when $D \propto 2^n$
- **Area law** The entanglement of a subregion of space scales as its area for a TNS
- **Efficiency** [gapped] Matrix Product States approximate well the ground states of gapped systems in 1 spatial dimension
- **Efficiency** [critical] Multi-scale Entanglement Renormalization Ansatz (MERA) approximate well the ground states of critical systems in 1 spatial dimension.
- **Symmetries** Physical symmetries can be implemented locally on the bond space
- **Inverse problem** TNS are the ground state of a local parent Hamiltonian

Successes and limits

Successes

- ♡ Arbitrary precision for $1d$ quantum systems
- ♡ Classification of topological phases in $1d$ and $2d$
- ♡ Progress on non-Abelian lattice Gauge theories
- ♡ AdS/CFT toy models

Limits

- ♠ Hard to contract in $d \geqslant 2$
- ♠ No continuum limit in $d \geqslant 2$
- ♠ Lack of analytic techniques

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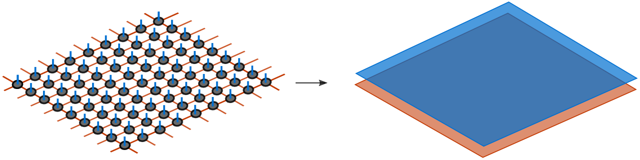
Can one apply tensor network techniques directly in the continuum, to QFT?



Lots of “Continuous tensor network” concepts

Tensor networks for quantum states $|\psi\rangle$

- 
  MPS $\rightarrow$ cMPS
  [Verstraete & Cirac 2010]

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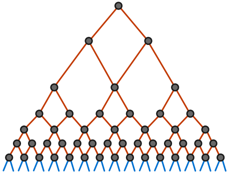
- 

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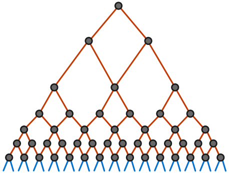
- 
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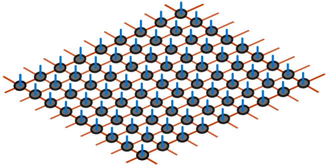
- 
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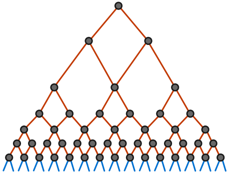
- 
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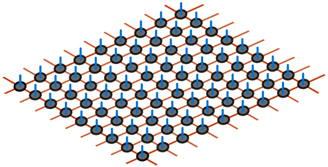
- 
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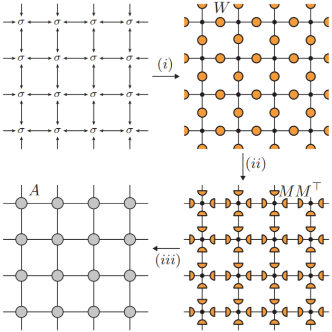
Tensor networks for partition functions $Z(\beta)$

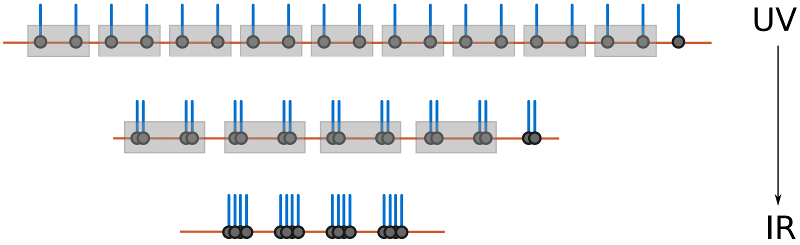
- StatMech in d
- Euclidean quantum in $d+1$



[Qi Hu et al. 2018]

Continuous Matrix Product States (cMPS)

Taking the continuum limit of a MPS



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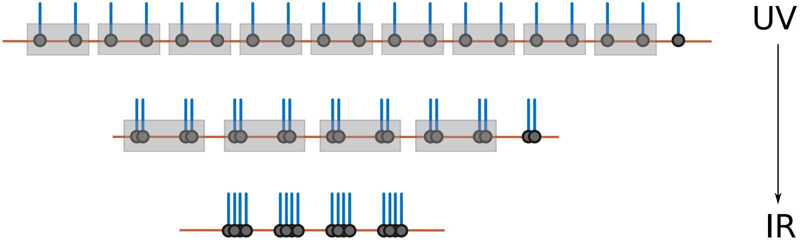

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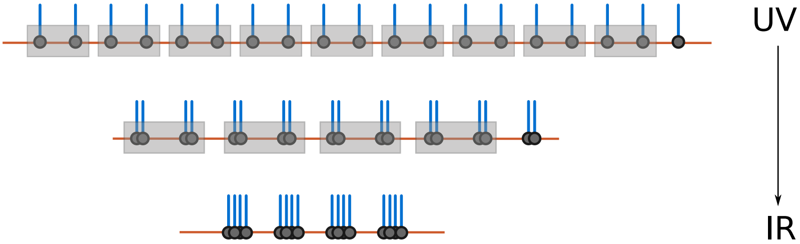


- the bond dimension D stays fixed
- the local physical dimension explodes $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \longrightarrow \mathscr{F}(L^2([x, x+dx]))$.

$\Longrightarrow$ **Spins** become **fields** – ($\simeq$ central limit theorem $\simeq$)

Continuous Matrix Product States

Type of ansatz for bosons on a fine grained $d = 1$ lattice

- Matrices $A_{i_k}(x)$ where the index i_k corresponds to $\psi^{\dagger i_k}(x)|0\rangle$ in physical space.

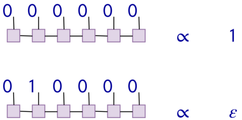
Informal cMPS definition

$$A_0 = \mathbb{1} + \varepsilon Q$$
$$A_1 = \varepsilon R$$
$$A_2 = \frac{(\varepsilon R)^2}{\sqrt{2}}$$
$$\dots$$
$$A_n = \frac{(\varepsilon R)^n}{\sqrt{n}}$$
$$\dots$$

so we go from ∞ to 2 matrices

Fixed by:

- Finite particle number



- Consistency

Continuous Matrix Product States

Definition

$$|Q, R, \omega\rangle = \langle\omega_L|\mathcal{P}\exp\left\{\int_0^L dx\; Q\otimes\mathbb{1} + R\otimes\psi^\dagger(x)\right\}|\omega_R\rangle\;|0\rangle$$

- Q, R are $D\times D$ matrices,
- $|\omega_L\rangle$ and $|\omega_R\rangle$ are boundary vectors $\in\mathbb{C}^D$, for p.b.c. $\langle\omega_L|\cdot|\omega_R\rangle\rightarrow \mathrm{tr}[\cdot]$
- $[\psi(x), \psi^\dagger(y)] = \delta(x-y)$

Idea:

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Idea:

$$\begin{aligned} A(x) &\simeq A_0 \mathbb{1} + A_1 \psi^\dagger(x) \\ &\simeq \mathbb{1} \otimes \mathbb{1} + \varepsilon Q \otimes \mathbb{1} + \varepsilon R \otimes \psi^\dagger(x) \\ &\simeq \exp\left[\varepsilon \left(Q \otimes \mathbb{1} + R \otimes \psi^\dagger(x) \right) \right] \end{aligned}$$

Computations

Some correlation functions

$$\left\langle \hat{\psi}(x)^\dagger \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{TL}(R \otimes \overline{R})\right]$$

$$\left\langle \hat{\psi}(x)^\dagger \hat{\psi}(0)^\dagger \hat{\psi}(0) \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{T(L-x)}(R \otimes \overline{R}) e^{Tx}(R \otimes \overline{R})\right]$$

$$\left\langle \hat{\psi}(x)^\dagger \left[-\frac{d^2}{dx^2}\right] \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{TL}([Q, R] \otimes [\overline{Q}, \overline{R}])\right]$$

with $T = Q \otimes \mathbb{1} + \mathbb{1} \otimes \bar{Q} + R \otimes \bar{R}$

Example

Lieb-Liniger Hamiltonian

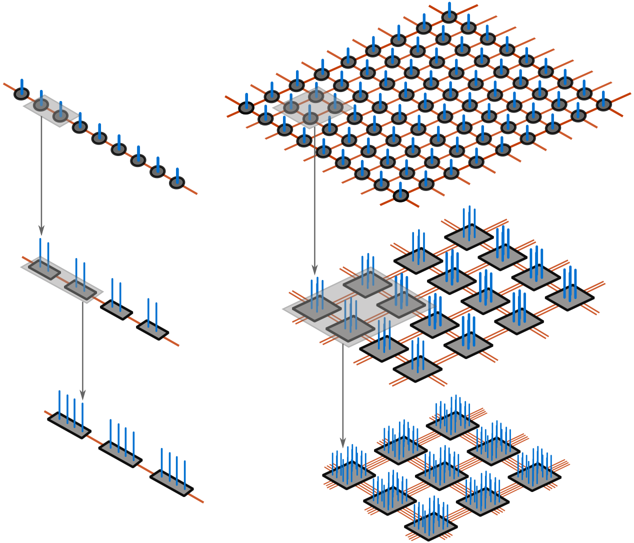
$$\mathcal{H} = \int_{-\infty}^{+\infty} dx \left[\frac{d\hat{\psi}^\dagger(x)}{dx} \frac{d\hat{\psi}(x)}{dx} + c\hat{\psi}^\dagger(x)\hat{\psi}^\dagger(x)\hat{\psi}(x)\hat{\psi}(x) \right]$$

Solve by **minimizing**:

$$\langle Q, R| \mathcal{H} |Q, R\rangle = f(Q, R)$$

with fixed particle density $\langle Q, R| \psi^\dagger(x)\psi(x) |Q, R\rangle$.

Continuous Tensor Networks: blocking



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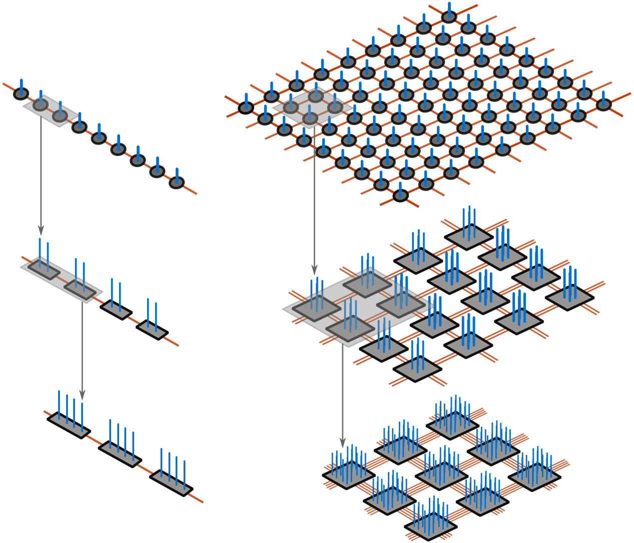

Upon blocking:

- The **physical** Hilbert space dimension *d* increases (idem cMPS $\Longrightarrow$ physical field)
- The **bond** dimension *D* increases too

Choice of trivial tensor

For **MPS**, not much choice:

 $=$ $+ \quad \varepsilon \cdots$

$$= \mathbb{1} \otimes |0\rangle + \varepsilon\, Q \otimes |0\rangle + \varepsilon\, R \otimes \psi^\dagger(\mathbf{x})|0\rangle$$

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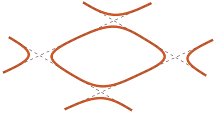
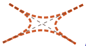
1. Take a δ between all legs $\sim$ GHZ state $T^{(0)} =$ 
   $\Longrightarrow$ trivial geometry
2. Take two identities $T^{(0)} =$ 
   $\Longrightarrow$ breakdown of Euclidean invariance
3. Take the sum of pairs of identities in both directions $T^{(0)} =$  $+$ 



Ansatz

1 – Take a "Trivial" tensor:

$$T^{(0)}_{\phi(1),\phi(2),\phi(3),\phi(4)} =$$



$$\sim \exp\left\{ \frac{-1}{2} \sum_{k=1}^{D} [\phi_k(1) - \phi_k(2)]^2 + [\phi_k(2) - \phi_k(3)]^2 + [\phi_k(3) - \phi_k(4)]^2 + [\phi_k(4) - \phi_k(1)]^2 \right\}$$

The indices ϕ are in $\mathbb{R}^D$ (and **not** $1, \cdots, D$)

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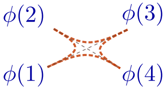

$$\sim \exp\left\{ \frac{-1}{2} \sum_{k=1}^{D} [\phi_k(1) - \phi_k(2)]^2 + [\phi_k(2) - \phi_k(3)]^2 + [\phi_k(3) - \phi_k(4)]^2 + [\phi_k(4) - \phi_k(1)]^2 \right\}$$

The indices ϕ are in $\mathbb{R}^D$ (and **not** $1, \cdots, D$)

2 – And add a "correction":

$$\exp\left\{ -\varepsilon^2 V[\phi(1), \cdots, \phi(4)] + \varepsilon^2 \alpha [\phi(1), \cdots, \phi(4)]\, \psi^\dagger(x) \right\}$$

Ansatz

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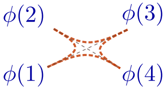

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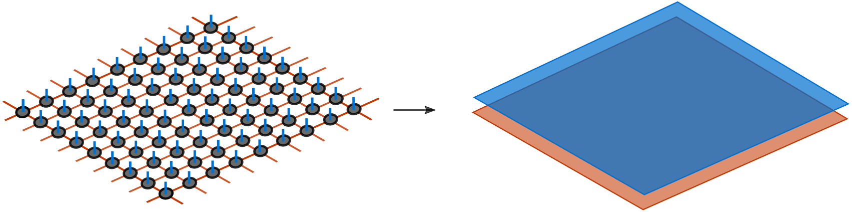
3 – Realize tensor contraction = functional integral and trivial tensor gives free field measure.

Functional integral definition



$$|V,\alpha\rangle = \int \mathcal{D}\phi \, \exp\left\{ -\int_\Omega \mathrm{d}^d x \, \frac{1}{2} \sum_{k=1}^{D} [\nabla\phi_k(x)]^2 + V[\phi(x)] - \alpha[\phi(x)] \, \hat{\psi}^\dagger(x) \right\} |0\rangle$$

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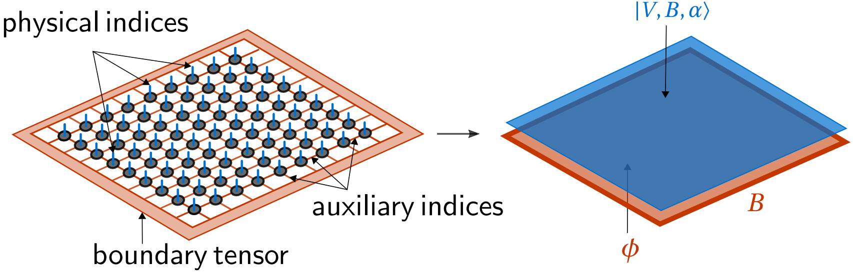
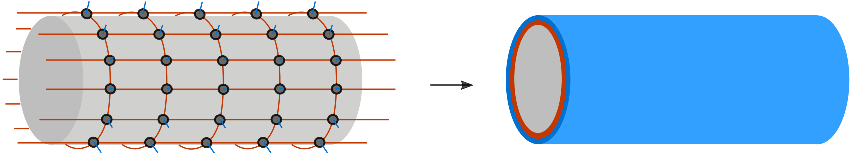


$$|V, B, \alpha\rangle = \int \mathcal{D}\phi \, B(\phi|_{\partial\Omega}) \exp\left\{-\int_{\Omega} \mathrm{d}^d x \, \frac{1}{2} \sum_{k=1}^{D} [\nabla \phi_k(x)]^2 + V[\phi(x)] - \alpha[\phi(x)] \, \psi^\dagger(x)\right\} |0\rangle$$

Operator definition



$$|V,\alpha\rangle =$$

$$\mathrm{tr}\left[\mathcal{T}\exp\left(-\int_0^T d\tau \int_S d\mathbf{x}\, \frac{\hat{\pi}_k(\mathbf{x})\hat{\pi}_k(\mathbf{x})}{2} + \frac{\nabla\hat{\phi}_k(\mathbf{x})\nabla\hat{\phi}_k(\mathbf{x})}{2} + V[\hat{\phi}(\mathbf{x})] - \alpha[\hat{\phi}(\mathbf{x})]\,\psi^\dagger(\tau,\mathbf{x})\right)\right]|0\rangle$$

where:

- $\hat{\phi}_k(\mathbf{x})$ and $\hat{\pi}_k(\mathbf{x})$ are k independent canonically conjugated pairs of (auxiliary) field *operators*: $[\hat{\phi}_k(\mathbf{x}), \hat{\phi}_l(\mathbf{y})] = 0$, $[\hat{\pi}(\mathbf{x})_k, \hat{\pi}_l(\mathbf{y})] = 0$, and $[\hat{\phi}_k(\mathbf{x}), \hat{\pi}_l(\mathbf{y})] = i\delta_{k,l}\,\delta(\mathbf{x}-\mathbf{y})$ acting on a space of $d-1$ dimensions.

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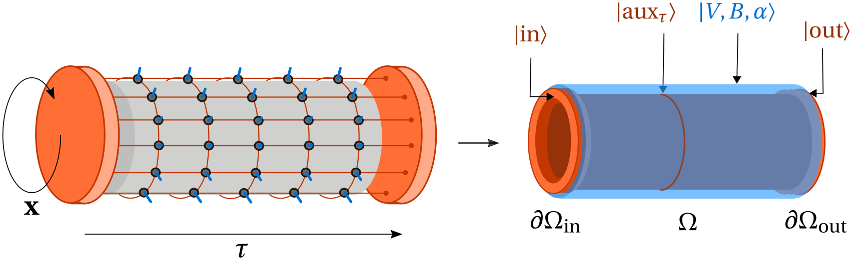

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Wave-function definition

A generic state $|\Psi\rangle$ in Fock space can be written:

$$|\Psi\rangle = \sum_{n=0}^{+\infty} \int_{\Omega^n} \frac{\varphi_n(x_1, \cdots, x_n)}{n!} \psi^\dagger(x_1) \cdots \psi^\dagger(x_n) |0\rangle$$

where ϕ_n is a symmetric n-particle wave-function

Functional integral representation

$$\varphi_n(x_1, \cdots, x_n) = \langle\, \alpha[\phi(x_1)] \cdots \alpha[\phi(x_n)]\, \rangle_{\text{aux}}$$

with:

$$\langle\cdot\rangle_{\text{aux}} = \int \mathcal{D}\phi \;\cdot\; B(\phi|_{\partial\Omega}) \exp\left[-\frac{1}{2}\int_{\Omega} \mathrm{d}^d x \,[\nabla\phi_k(x)]^2 + V[\phi(x)]\right]$$

- $\sim$ Moore-Read wave-function for Quantum Hall, but generic QFT

Expressivity and stability

How big are cTNS?

Stability

The sum of two cTNS of bond field dimension D_1 and D_2 is a cTNS with bond field dimension $D \leqslant D_1 + D_2 + 1$:

$$|V_1, \alpha_1\rangle + |V_2, \alpha_2\rangle = |W, \beta\rangle$$

Expressiveness

All states in the Fock space can be approximated by cTNS:

- A field coherent state is a cTNS with $D = 0$
- Stability allows to get all sums of field coherent states

Note: expressiveness can also be obtained with $D = 1$ but it is less natural. Flexibility in D makes the expressivity higher for restricted classes of V and α.

Computations

Define generating functional for normal ordered correlation functions

$$\mathcal{Z}_{j',j} = \frac{1}{\langle V, \alpha | V, \alpha \rangle} \langle V, \alpha | \exp\left(\int \mathrm{d}x\, j'(x) \psi^\dagger(x) \right) \exp\left(\int \mathrm{d}x\, j(x) \psi(x) \right) | V, \alpha \rangle$$

Operator representation

$$\mathcal{Z}_{j'j} = \operatorname{tr}\left[B \otimes B^* \mathcal{T} \exp\left\{ \int_{-T/2}^{T/2} \left(\mathrm{T}_{j'j} - \int_S j \cdot j' \right) \right\} \right]$$

with **transfer matrix**:

$$\mathrm{T}_{j'j} = \int_S \mathbf{dx}\, \mathcal{H}(\mathbf{x}) \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{H}^*(\mathbf{x}) + \left(\alpha[\hat{\phi}(\mathbf{x})] + j'(\mathbf{x}) \right) \otimes \left(\alpha[\hat{\phi}(\mathbf{x})]^* + j(\mathbf{x}) \right)$$

and

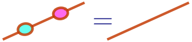
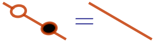
$$\mathcal{H}(\mathbf{x}) = \sum_{k=1}^{D} \frac{[\hat{\pi}_k(\mathbf{x})]^2 + \left[\nabla \hat{\phi}_k(\mathbf{x}) \right]^2}{2} + V[\hat{\phi}(\mathbf{x})]$$

$\Longrightarrow$ cMPS brought us from 1 to 0, cTNS bring us from d to $d-1$.

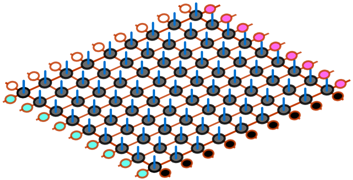
Redundancies

Discrete redundancy

Different elementary tensors are **equivalent**, they give the same state:

when  and 

up to **boundary** terms:



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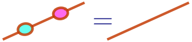
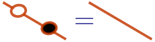
when  and 

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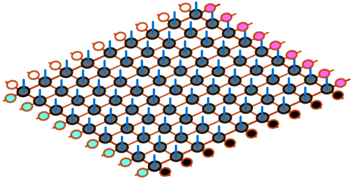
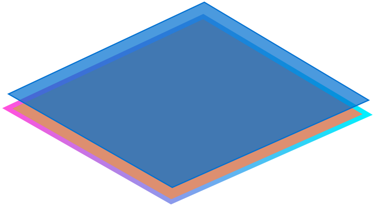


Continuum redundancy

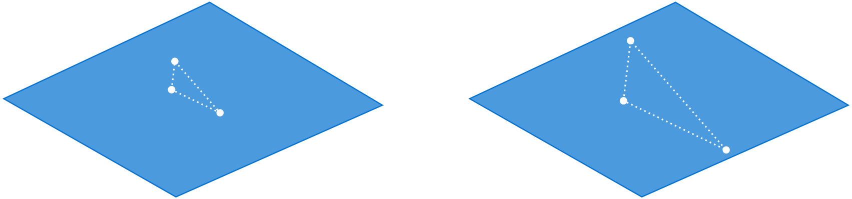
$$V(\phi) \to V(\phi) + \nabla \cdot \mathscr{F}[x, \phi(x)]$$

Just Stokes' theorem. If Ω has a boundary $\partial\Omega$:

$$\mathcal{D}[\phi] \to \mathcal{D}[\phi] \exp\left\{ \oint_{\partial\Omega} \mathrm{d}^{d-1}x \, \mathscr{F}[x, \phi(x)] \cdot \mathbf{n}(x) \right\}$$



Rescaling



$$C(x_1, \cdots, x_n) = \langle T(1) | \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) | T(1) \rangle,$$

the objective is to find a tensor $T(\lambda)$ of new parameters such that:

$$C(\lambda x_1, \cdots, \lambda x_n) \propto \langle T(\lambda) | \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) | T(\lambda) \rangle.$$

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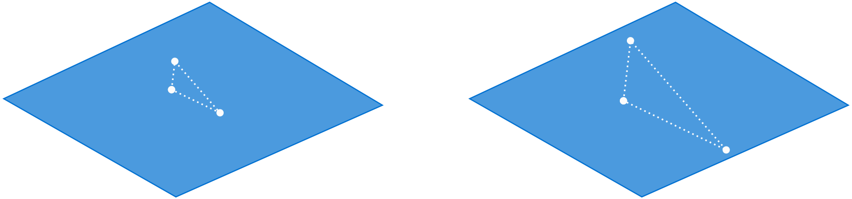

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Doable exactly:

$$V \to \lambda^d V \circ \lambda^{\frac{2-d}{2}} \quad \text{and} \quad \alpha \to \lambda^{\frac{d}{2}} \alpha \circ \lambda^{\frac{2-d}{2}}$$

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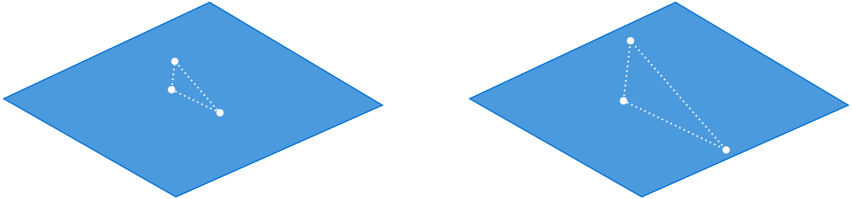

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- $d = 2$, All powers of the field in V and α yield relevant couplings
- $d = 3$, The powers $p = 1, 2, 3, 4, 5$ of the field in V yield relevant $\Delta > 0$ couplings. $p = 6$ is marginal in V. For α, $p = 1, 2$ are relevant and $p = 3$ is marginal. All other p are irrelevant.

Renormalization

Scaling

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For finite bond field dimension in $d = 3$, finite number of parameters for **renormalized** cTNS:

$$V(\phi) = A\phi + B\phi\phi + C\phi\phi\phi + D\phi\phi\phi\phi + E\phi\phi\phi\phi\phi + F\phi\phi\phi\phi\phi\phi$$
$$\alpha(\phi) = X\phi + Y\phi\phi + Z\phi\phi\phi$$

Proper renormalization procedure not checked yet

Getting back cMPS

One can get back cMPS with finite bond dimension by:

1. **Compactification** Take $d-1$ dimensions out of d to be very small

$$|V,B,\alpha\rangle \simeq \mathrm{tr}\left[\hat{B}\,\mathcal{T}\exp\left(-\int_0^T \mathrm{d}\tau \sum_{k=1}^{D} \frac{\hat{P}_k^2}{2} + V[\hat{X}] - \alpha[\hat{X}]\,\psi^\dagger(\tau)\right)\right]|0\rangle$$

$\Longrightarrow$ Hilbert space of a quantum particle in D space dimensions.

2. **Quantization** Take V with D deep minima to force the auxiliary field to take only D possibilities

Generalization

For a general Riemanian manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, define:

$$|V, B, \alpha\rangle = \int \mathcal{D}\phi\, B(\phi|_{\partial\mathcal{M}}) \exp\left\{-\int_{\mathcal{M}} \mathrm{d}^d x \sqrt{g}\left(\frac{g^{\mu\nu}\partial_\mu\phi_k\partial_\nu\phi_k}{2} + V[\phi, \nabla\phi] - \alpha[\phi, \nabla\phi]\,\psi^\dagger\right)\right\}|0\rangle$$

i.e. add curvature and possible anisotropies in V and α

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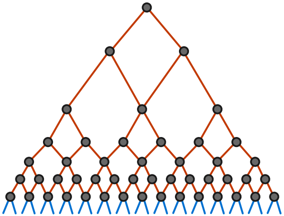
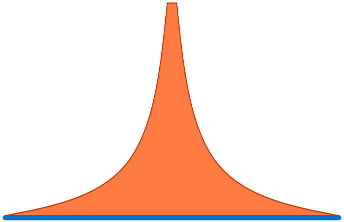
Example: $\alpha[x, \phi, \nabla\phi]$ localized on the boundary and hyberbolic metrix g:





$\rightarrow$ **cMERA**-like in $d-1$ dimensions

Future

Limitations and work for the future

- Quite formal out of the Gaussian regime
- Computation through dimensional reduction not trivial
- Limited to bosonic field theories (so far)
- Gauge invariant states
- Can one say anything about topology?

Summary

$$|V, B, \alpha\rangle = \int \mathcal{D}\phi \, B(\phi|_{\partial\Omega}) \exp\left\{-\int_{\Omega} \mathrm{d}^d x \, \frac{1}{2}\sum_{k=1}^{D} [\nabla\phi_k(x)]^2 + V[\phi(x)] - \alpha[\phi(x)]\, \psi^\dagger(x)\right\} |0\rangle$$

Continuous tensor network states are natural continuum limits of tensor network states and natural higher d extensions of continuous matrix product states.

1. Obtained from discrete tensor networks
2. Can be made Euclidean invariant
3. **Motto of tensor networks**: trade a dimension for a variational optimization
4. Still need to be properly renormalized (in perturbative and RG sense)
5. Still needs to be used to approximate non-trivial non-Gaussian ground states