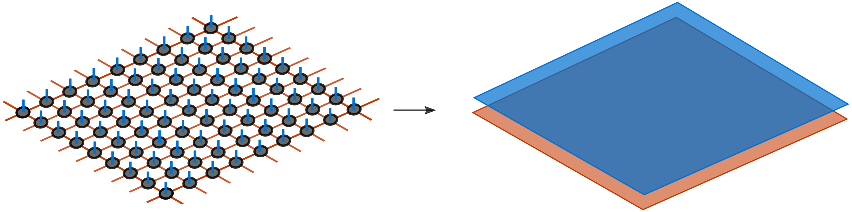
Continuous quantum measurements and tensor networks

quantum information tools brought to the continuum

Antoine Tilloy

Max Planck Institute of Quantum Optics, Garching, Germany



Séminaire

Institut de Physique Théorique, Saclay, France

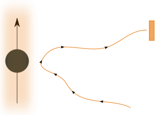
April 15th, 2019

Alexander von Humboldt Stiftung/Foundation

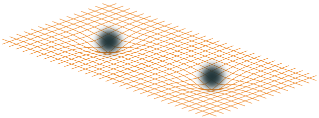
MPQ

3 questions in quantum mechanics

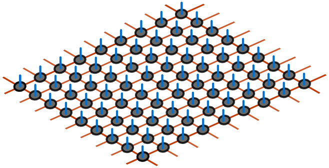
Continuous measurement
How to gently measure and control quantum systems?

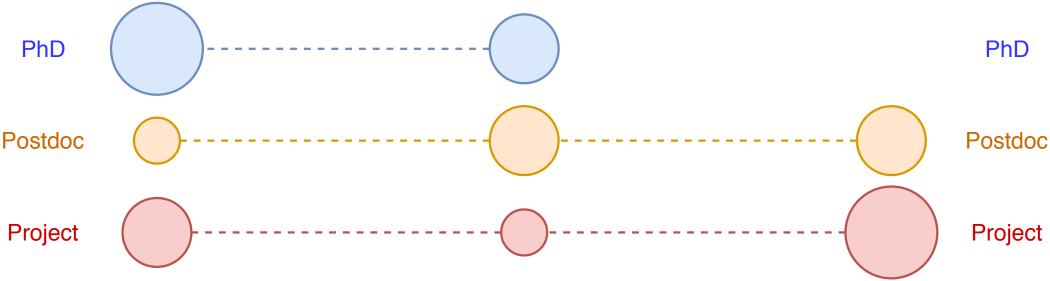


Gravity and quantum
Could gravity, in principle, not be quantum?

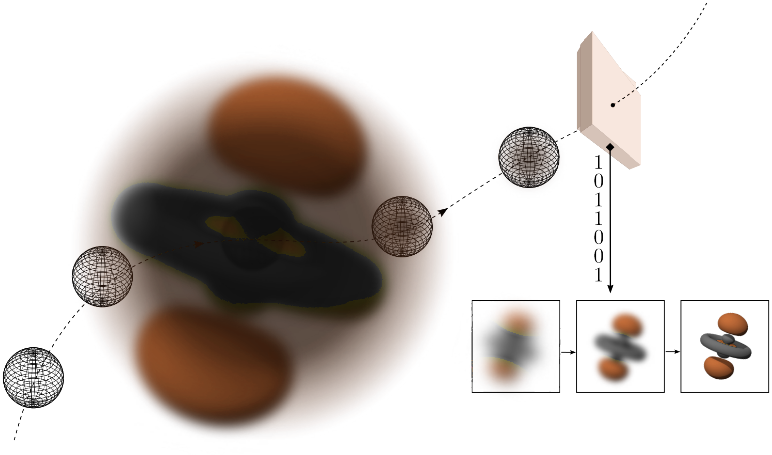


Many-body & QFT
How to efficiently parameterize many-body and QFT states





Observation



Motivation

"We know that the moon is demonstrably not there when nobody looks"

David Mermin 1981

Introduction

Measurement postulate

For a system "described" by $|\psi\rangle \in \mathscr{H}$ and a measurement of projectors Π_i such that $\sum_i \Pi_i = \mathbb{1}$:

- ♣ **Born rule** : Result i with probability $\mathbb{P}[i] = \langle\psi|\Pi_i|\psi\rangle$
- ♣ **Collapse** : $|\psi\rangle \longrightarrow \frac{\Pi_i|\psi\rangle}{\sqrt{\mathbb{P}[i]}}$

Max Born

John von Neumann

Introduction

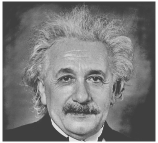
Measurement postulate

For a system "described" by $|\psi\rangle \in \mathscr{H}$ and a measurement of projectors Π_i such that $\sum_i \Pi_i = \mathbb{1}$:

- **Born rule** : Result i with probability $\mathbb{P}[i] = \langle\psi|\Pi_i|\psi\rangle$
- **Collapse** : $|\psi\rangle \longrightarrow \frac{\Pi_i|\psi\rangle}{\sqrt{\mathbb{P}[i]}}$


Max Born


John von Neumann


Albert Einstein


John S. Bell

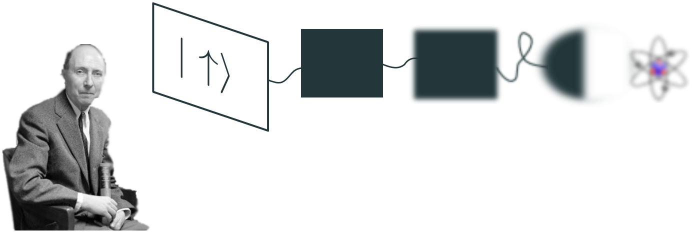
What is a measurement?

- When can the postulate be applied?
- Can measurement be deduced from other postulates?

Introduction

Moving the Heisenberg cut

Limit between the **system**, obeying the Schrödinger equation and the **observer** who can apply the measurement postulate.

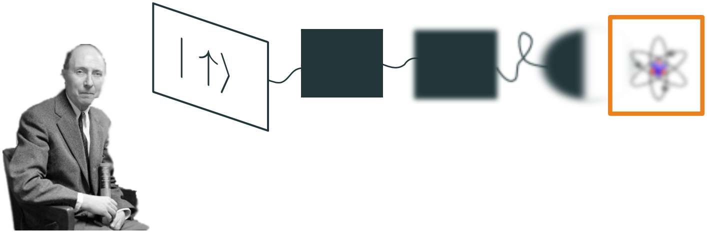


Eugene Wigner

Introduction

Moving the Heisenberg cut

Limit between the **system**, obeying the Schrödinger equation and the **observer** who can apply the measurement postulate.



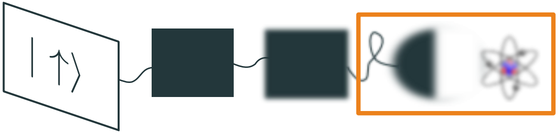
Eugene Wigner

Introduction

Moving the Heisenberg cut

Limit between the **system**, obeying the Schrödinger equation and the **observer** who can apply the measurement postulate.

Eugene Wigner

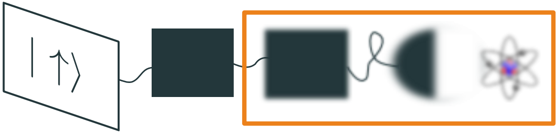


Introduction

Moving the Heisenberg cut

Limit between the **system**, obeying the Schrödinger equation and the **observer** who can apply the measurement postulate.

Eugene Wigner

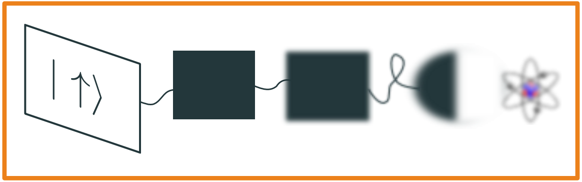


Introduction

Moving the Heisenberg cut

Limit between the **system**, obeying the Schrödinger equation and the **observer** who can apply the measurement postulate.

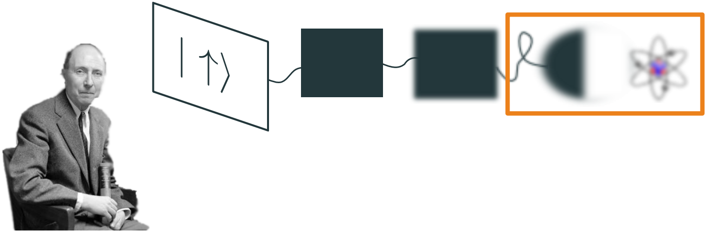
Eugene Wigner



Introduction

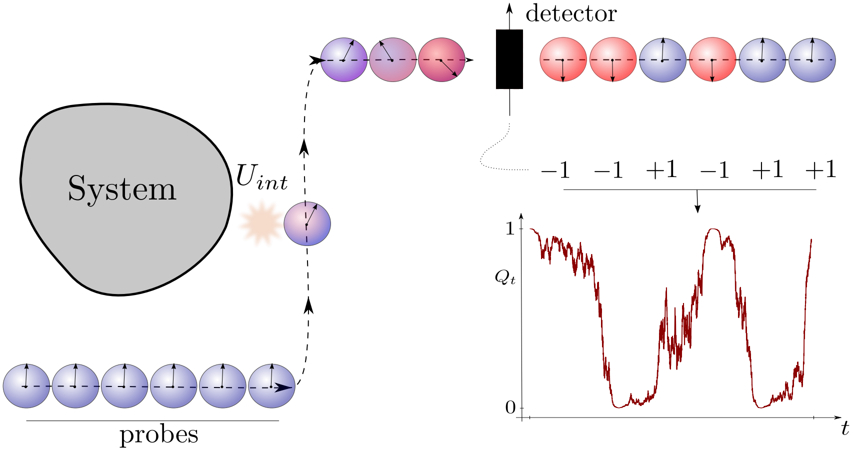
Moving the Heisenberg cut

Limit between the **system**, obeying the Schrödinger equation and the **observer** who can apply the measurement postulate.



Eugene Wigner

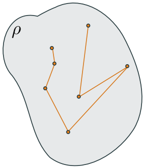
Continuous observation



Repeated interactions

Discrete quantum trajectories

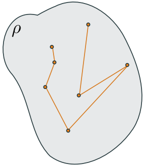
A sequence of $|\psi_n\rangle$ or ρ_n (random) and the corresponding measurement results $\delta_n = \pm 1$.



Repeated interactions

Discrete quantum trajectories

A sequence of $|\psi_n\rangle$ or ρ_n (random) and the corresponding measurement results $\delta_n = \pm 1$.

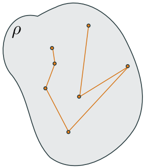


- Make the interaction between system and probe smoother $U_{\text{int}} = \mathbb{1} + i\sqrt{\varepsilon}\, A_{\text{sys}} \otimes B_{\text{probe}}$
- Increase the frequency at which probes are sent: $\tau \propto \varepsilon$

Repeated interactions

Discrete quantum trajectories

A sequence of $|\psi_n\rangle$ or ρ_n (random) and the corresponding measurement results $\delta_n = \pm 1$.



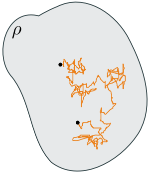
- Make the interaction between system and probe smoother $U_{\text{int}} = \mathbb{1} + i\sqrt{\varepsilon}\, A_{\text{sys}} \otimes B_{\text{probe}}$
- Increase the frequency at which probes are sent: $\tau \propto \varepsilon$

Continuous quantum trajectories

A continuous map $|\psi_t\rangle$ or ρ_t (random) and the corresponding continuous measurement signal $y_t \propto \sqrt{\varepsilon}\sum_k \delta_k$. Typically:

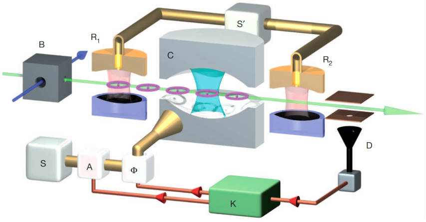
$$\mathrm{d}|\psi_t\rangle = \left[-iH\,\mathrm{d}t + \sqrt{\gamma}(A - \langle A\rangle)\mathrm{d}W_t - \frac{\gamma}{2}(A - \langle A\rangle)^2\,\mathrm{d}t\right]|\psi_t\rangle$$

where W_t Brownian ⚠ Essentially a central limit theorem result ⚠

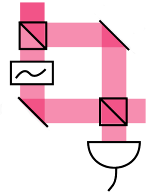


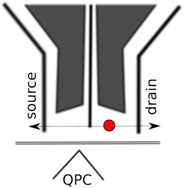
In practice

- Discrete situations "LKB style", with **actual** repeated interactions



- Almost "true" continuous measurement settings (quantum optics, quantum dots)



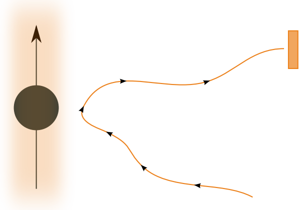


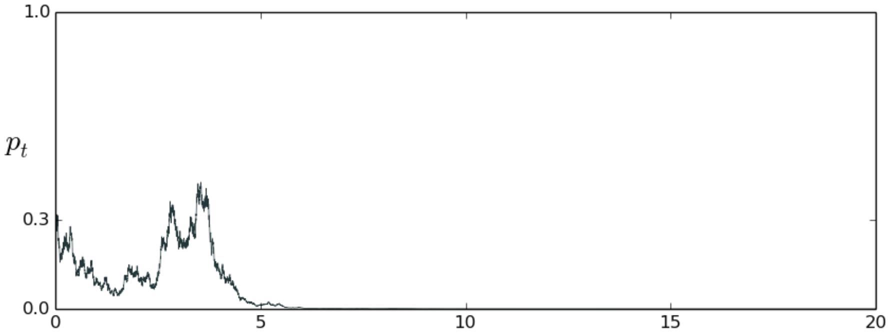
Example 0

Situation considered

Pure continuous measurement of a qubit:

- for the population: $p_t = |\langle\uparrow|\psi_t\rangle|^2$
- one can show: $\mathrm{d}p_t = \sqrt{\gamma}\, p_t(1 - p_t)\,\mathrm{d}W_t$



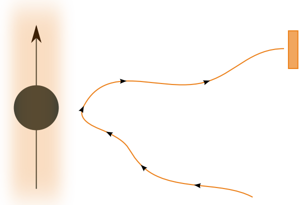


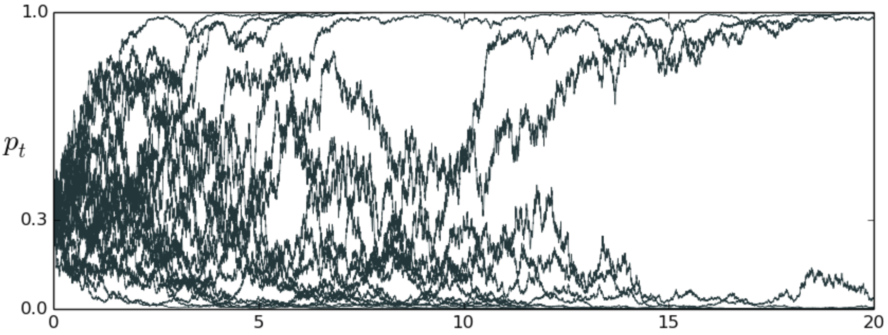
Example 0

Situation considered

Pure continuous measurement of a qubit:

- for the population: $p_t = |\langle\uparrow|\psi_t\rangle|^2$
- one can show: $\mathrm{d}p_t = \sqrt{\gamma}\, p_t(1-p_t)\,\mathrm{d}W_t$





Questions

Measurement is now **dynamical** with a time scale γ^{-1}. Hence one can:

- ♣ Optimize it
- ♣ Study its competition with (few-body) unitary dynamics $\propto \omega_i$
- ♣ Exploit it for real-time “soft” control

Questions

Measurement is now **dynamical** with a time scale γ^{-1}. Hence one can:

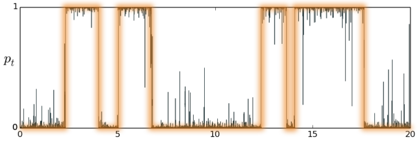
- ♣ Optimize it
- ♣ Study its competition with (few-body) unitary dynamics $\propto \omega_i$
- ♣ Exploit it for real-time "soft" control

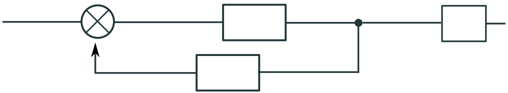
Strong continuous observation $\gamma \gg \omega_i$

- Non-demolition measurement
- Quantum jumps
- Quantum spikes

Weak continuous observation $\gamma \sim \omega_i$

- Optimization
- Control
- Continuous quantum error correction



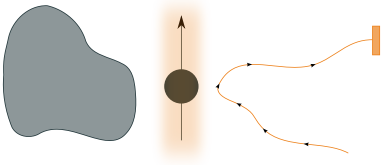


Strong measurement limit: example 1

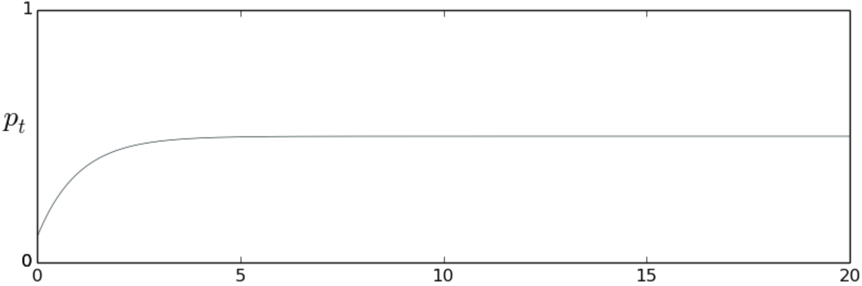
Situation considered

Qubit coupled to a thermal bath

- p_t ground state population
- Thermal bath $p_t \to p^{\text{Boltzmann}}$
- Continuous energy measurement $p_t \to 0$ or 1



No measurement, $\gamma = 0\,\lambda$

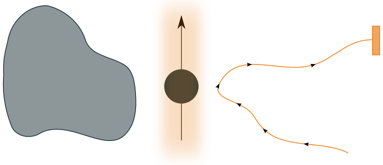


Strong measurement limit: example 1

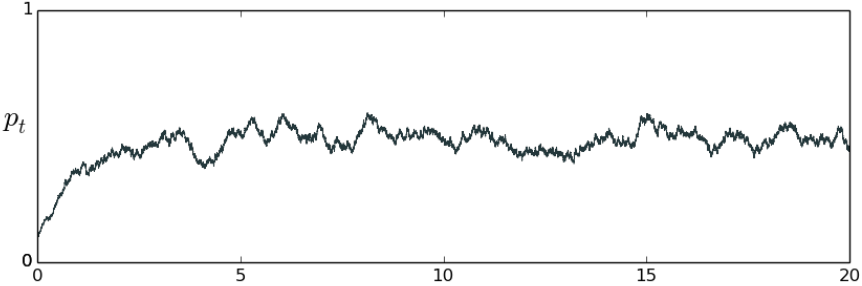
Situation considered

Qubit coupled to a thermal bath

- p_t ground state population
- Thermal bath $p_t \to p^{\text{Boltzmann}}$
- Continuous energy measurement $p_t \to 0$ or 1



Weak measurement, $\gamma = 0.1\,\lambda$

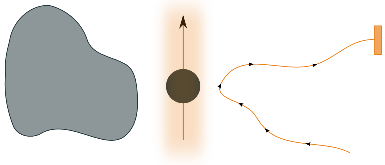


Strong measurement limit: example 1

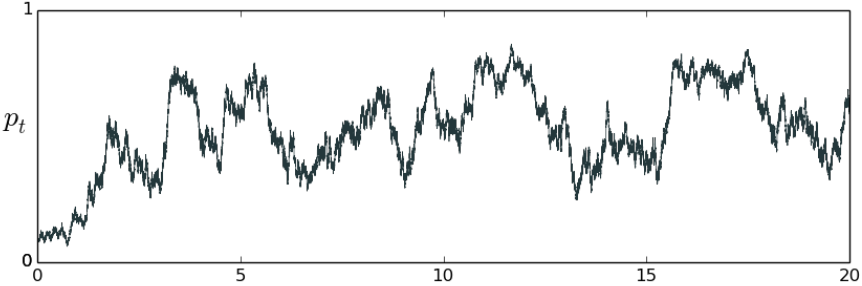
Situation considered

Qubit coupled to a thermal bath

- p_t ground state population
- Thermal bath $p_t \to p^{\text{Boltzmann}}$
- Continuous energy measurement $p_t \to 0$ or 1



Decent measurement, $\gamma = \lambda$

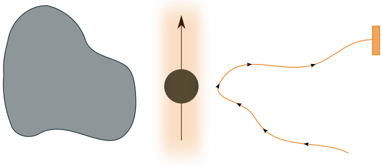


Strong measurement limit: example 1

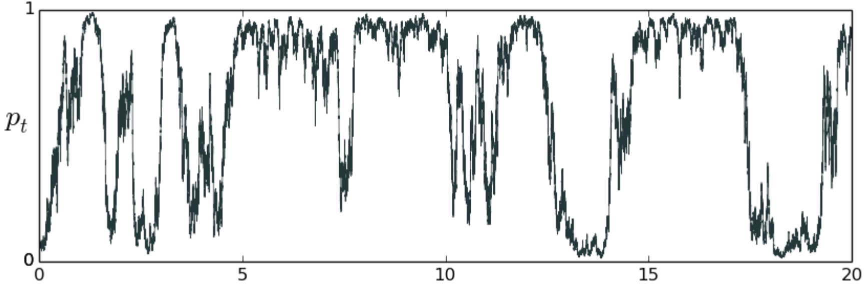
Situation considered

Qubit coupled to a thermal bath

- p_t ground state population
- Thermal bath $p_t \to p^{\text{Boltzmann}}$
- Continuous energy measurement $p_t \to 0$ or 1



Getting strong measurement, $\gamma = 10\,\lambda$

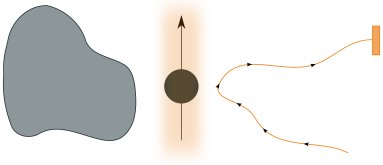


Strong measurement limit: example 1

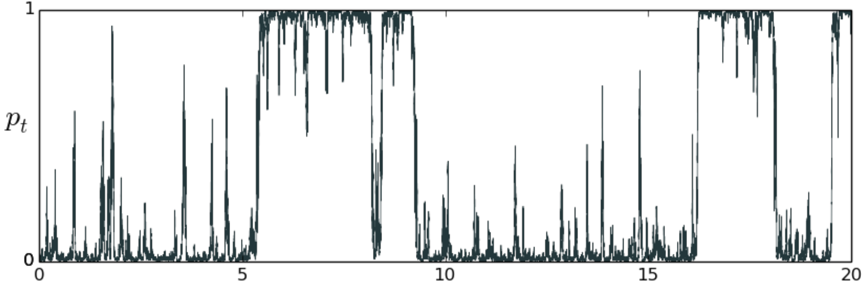
Situation considered

Qubit coupled to a thermal bath

- p_t ground state population
- Thermal bath $p_t \to p^{\text{Boltzmann}}$
- Continuous energy measurement $p_t \to 0$ or 1



Pretty strong measurement, $\gamma = 100\,\lambda$

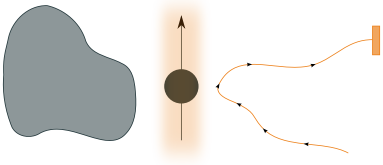


Strong measurement limit: example 1

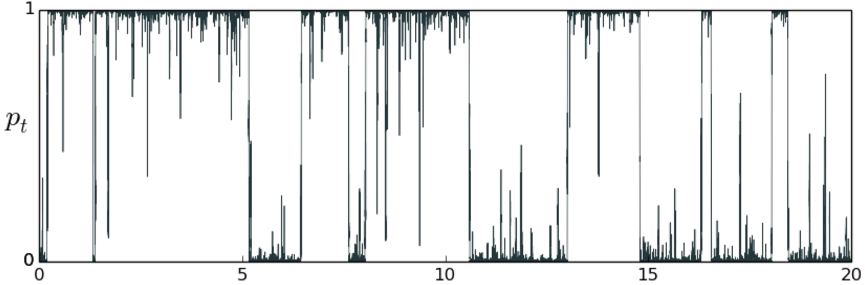
Situation considered

Qubit coupled to a thermal bath

- p_t ground state population
- Thermal bath $p_t \to p^{\text{Boltzmann}}$
- Continuous energy measurement $p_t \to 0$ or 1



Strong measurement, $\gamma = 1000\,\lambda$

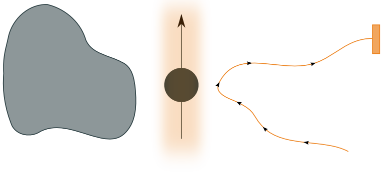


Strong measurement limit: example 1

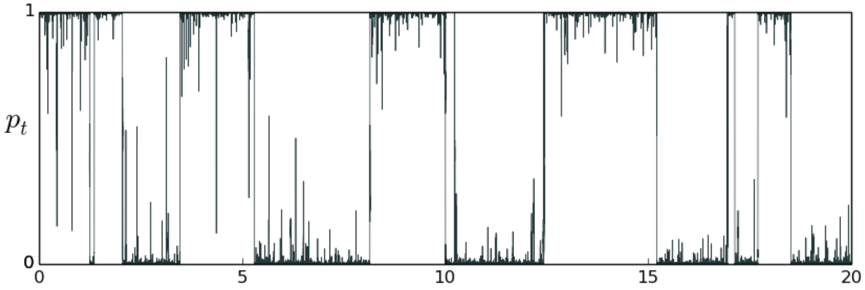
Situation considered

Qubit coupled to a thermal bath

- p_t ground state population
- Thermal bath $p_t \to p^{\text{Boltzmann}}$
- Continuous energy measurement $p_t \to 0$ or 1



Very strong measurement, $\gamma = 10^4\,\lambda$

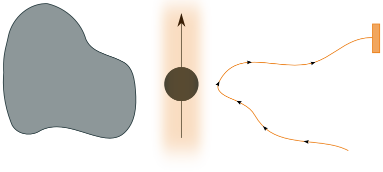


Strong measurement limit: example 1

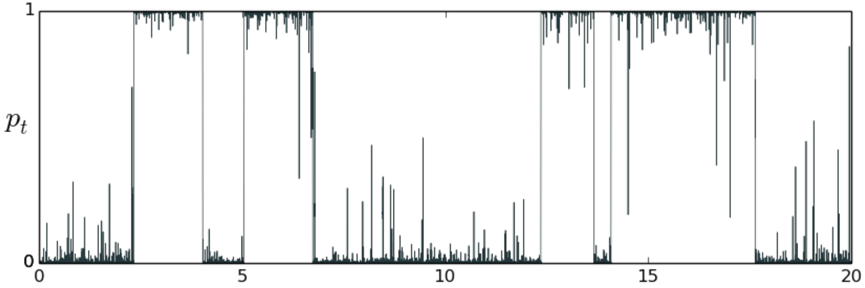
Situation considered

Qubit coupled to a thermal bath

- p_t ground state population
- Thermal bath $p_t \to p^{\text{Boltzmann}}$
- Continuous energy measurement $p_t \to 0$ or 1



Über strong measurement, $\gamma = 10^5\,\lambda$

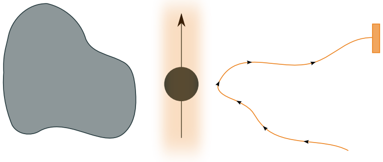


Strong measurement limit: example 1

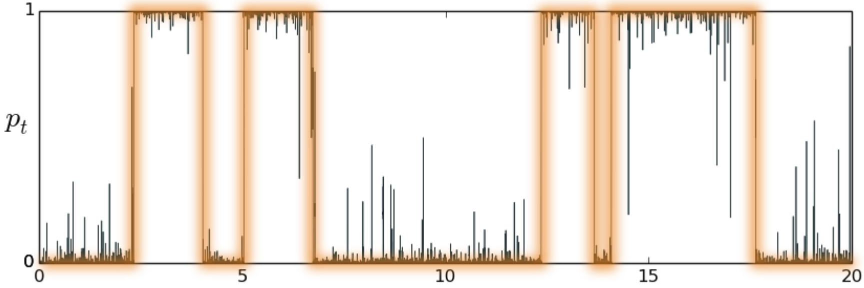
Situation considered

Qubit coupled to a thermal bath

- p_t ground state population
- Thermal bath $p_t \to p^{\text{Boltzmann}}$
- Continuous energy measurement $p_t \to 0$ or 1



Über strong measurement, $\gamma = 10^5\,\lambda$

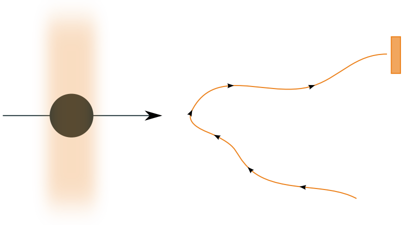


Strong measurement limit: example 2

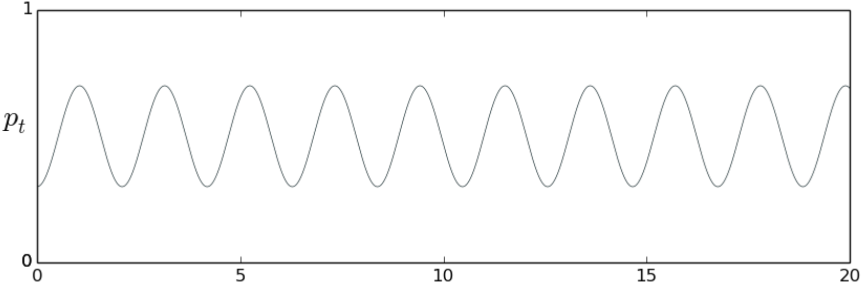
System considered

Qubit in a magnetic field $\perp$ measurement basis

- $p_t = |\langle\psi_t|\uparrow\rangle_z|^2$
- $H = \frac{\omega}{2}\sigma_x$: Rabi oscillations $p_t \sim \cos(\omega t)$
- Measurement $p_t \to 0$ or 1



No measurement, $\gamma = 0\,\omega$

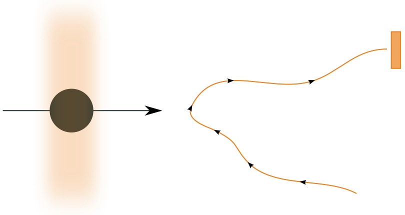


Strong measurement limit: example 2

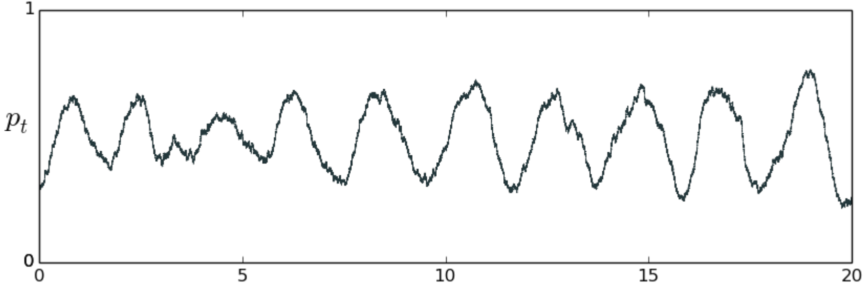
System considered

Qubit in a magnetic field $\perp$ measurement basis

- $p_t = |\langle\psi_t|\uparrow\rangle_z|^2$
- $H = \frac{\omega}{2}\sigma_x$: Rabi oscillations $p_t \sim \cos(\omega t)$
- Measurement $p_t \to 0$ or 1



Weak measurement, $\gamma = 0.1\,\omega$

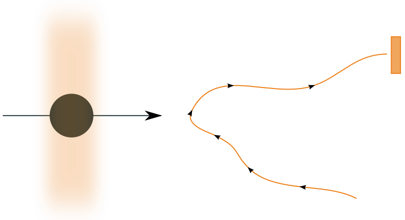


Strong measurement limit: example 2

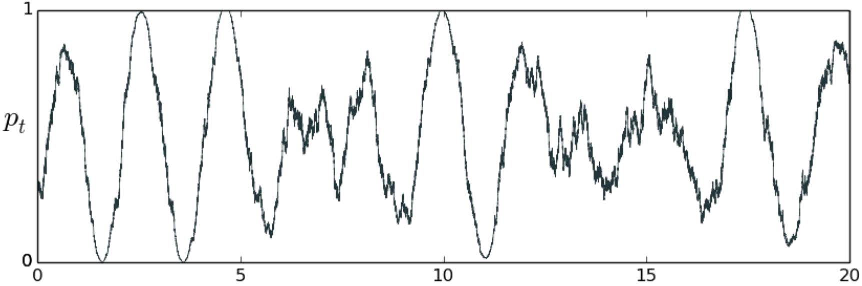
System considered

Qubit in a magnetic field $\perp$ measurement basis

- $p_t = |\langle\psi_t|\uparrow\rangle_z|^2$
- $H = \frac{\omega}{2}\sigma_x$: Rabi oscillations $p_t \sim \cos(\omega t)$
- Measurement $p_t \to 0$ or 1



Decent measurement, $\gamma = \omega$

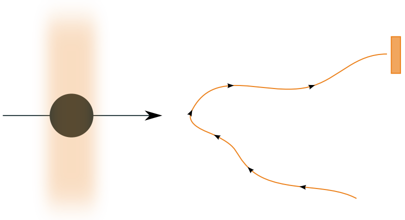


Strong measurement limit: example 2

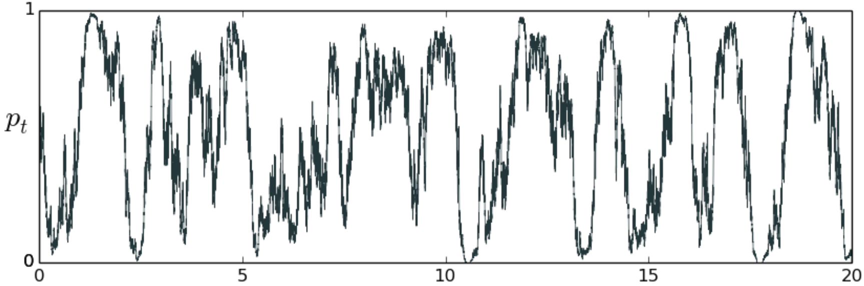
System considered

Qubit in a magnetic field $\perp$ measurement basis

- $p_t = |\langle \psi_t | \uparrow \rangle_z|^2$
- $H = \frac{\omega}{2}\sigma_x$: Rabi oscillations $p_t \sim \cos(\omega t)$
- Measurement $p_t \to 0$ or 1



Getting strong measurement, $\gamma = 10\,\omega$

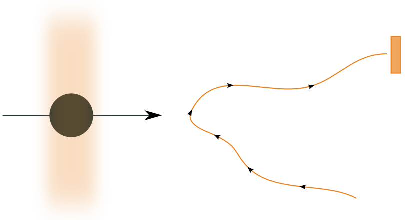


Strong measurement limit: example 2

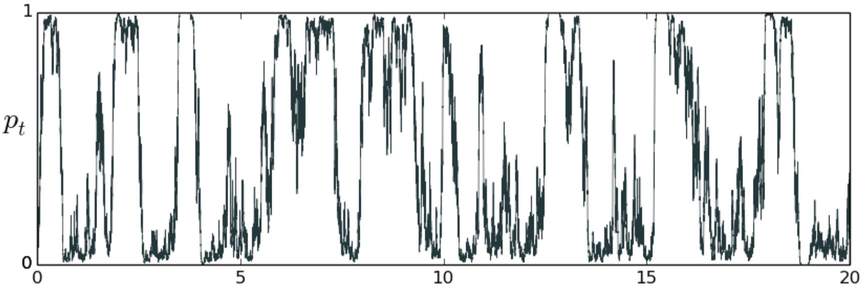
System considered

Qubit in a magnetic field $\perp$ measurement basis

- $p_t = |\langle\psi_t|\uparrow\rangle_z|^2$
- $H = \frac{\omega}{2}\sigma_x$: Rabi oscillations $p_t \sim \cos(\omega t)$
- Measurement $p_t \to 0$ or 1



Pretty strong measurement, $\gamma = 30\,\omega$

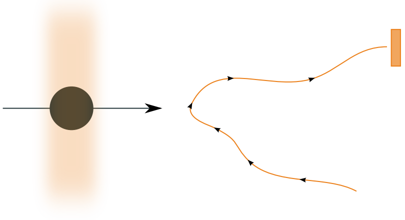


Strong measurement limit: example 2

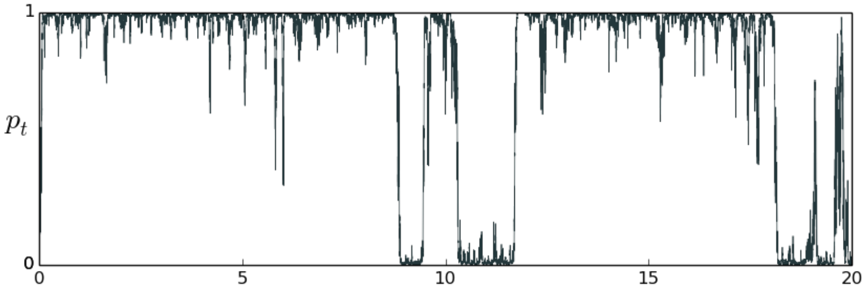
System considered

Qubit in a magnetic field $\perp$ measurement basis

- $p_t = |\langle\psi_t|\uparrow\rangle_z|^2$
- $H = \frac{\omega}{2}\sigma_x$: Rabi oscillations $p_t \sim \cos(\omega t)$
- Measurement $p_t \to 0$ or 1



Strong measurement, $\gamma = 100\,\omega$

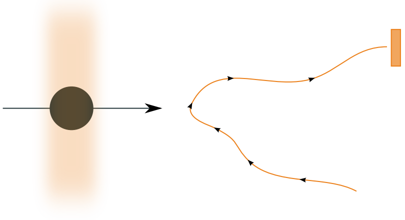


Strong measurement limit: example 2

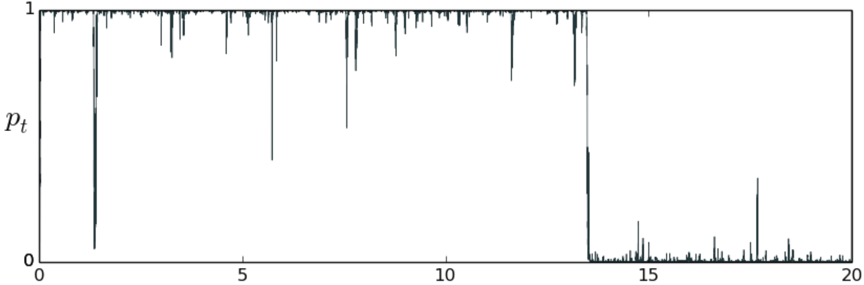
System considered

Qubit in a magnetic field $\perp$ measurement basis

- $p_t = |\langle \psi_t | \uparrow \rangle_z|^2$
- $H = \frac{\omega}{2}\sigma_x$: Rabi oscillations $p_t \sim \cos(\omega t)$
- Measurement $p_t \to 0$ or 1



Very strong measurement, $\gamma = 300\,\omega$

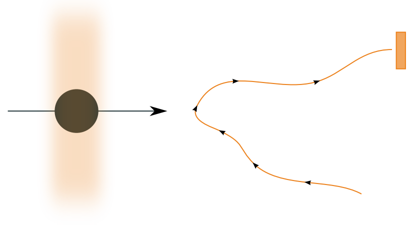


Strong measurement limit: example 2

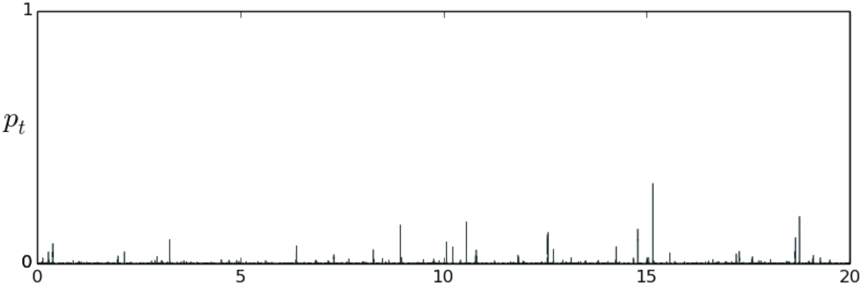
System considered

Qubit in a magnetic field $\perp$ measurement basis

- $p_t = |\langle\psi_t|\uparrow\rangle_z|^2$
- $H = \frac{\omega}{2}\sigma_x$: Rabi oscillations $p_t \sim \cos(\omega t)$
- Measurement $p_t \to 0$ or 1



Über strong measurement, $\gamma = 1000\,\omega$



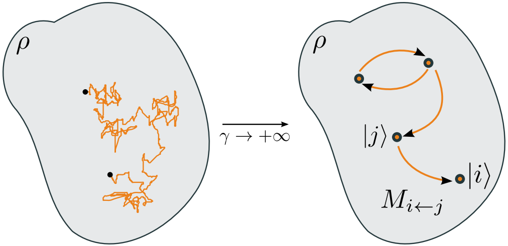
Theorem: jumps

1. Markovian evolution $\mathcal{L}(\rho_t) = L(\rho_t) - i\,[H, \rho_t]$
2. Continuous measurement of $\mathcal{O} = \sum_k \lambda_k |k\rangle\langle k|$

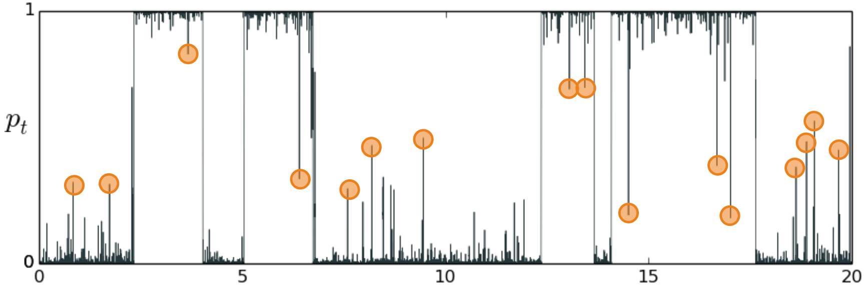
Quantum jumps

When $\gamma \to +\infty$, ρ_t converges to a **Markov chain** with transition matrix M:

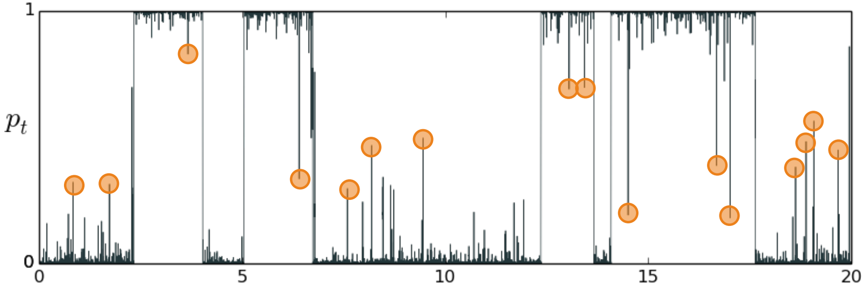
$$M_{i\leftarrow j} = \overbrace{L^{ii}_{jj}}^{\text{“incoherent” contribution}} + \underbrace{\frac{1}{4\gamma}\left|\frac{H_{ij}}{\lambda_i - \lambda_j}\right|^2}_{\text{“coherent” contribution}}$$



A subtlety: spikes



A subtlety: spikes



Spikes:

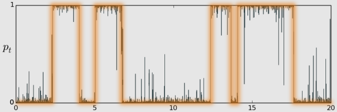
- Remain in the limit
- Are Levy distributed
- Are univeral
- Are experimentally relevant (e.g. for control)

Carrying computations rigorously, one discovers things people did not expect and thought were experimental mistakes

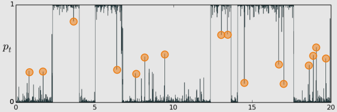
Some results

Strong continuous measurement

1. Jumps



2. Spikes



- M Bauer, D Bernard, AT JPA 2015
- AT, M Bauer, D Bernard PRA 2015
- M Bauer, D Bernard, AT JPA 2016

Others

1. Control
   - A T, M Bauer, D Bernard EPL 2014
2. Optimal measurement
   - AT, PRA 2016
3. Exact results
   - AT, PRA-Rapid 2018
4. Non-Markovian exploration
   - AT, Quantum 2017
5. Many-body exploration
   - X Cao, AT, A De Luca, 2018

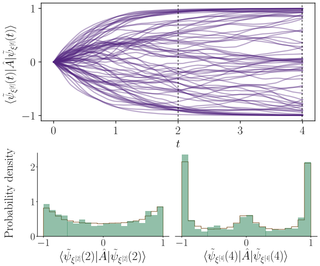
Future

Fast transition in the field in the last $2-3$ years: **new questions**

Non-Markovianity

How to include it in the theory?

- N-M feedback
- N-M measurement

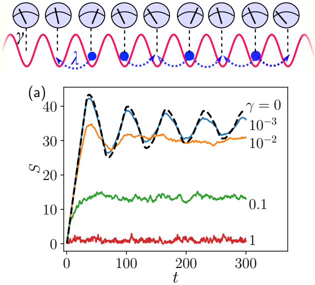


♠ `Non-Markovian Monte-Carlo`
AT, Quantum 2017

Many-body

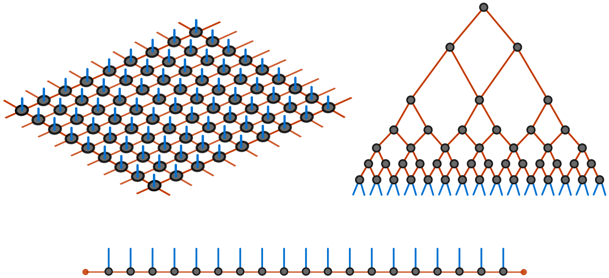
Joining measurement and MB dynamics

- For integrable models
- KPZ universality class?



♠ `arXiv:1804.04638`
X Cao, AT, A De Luca

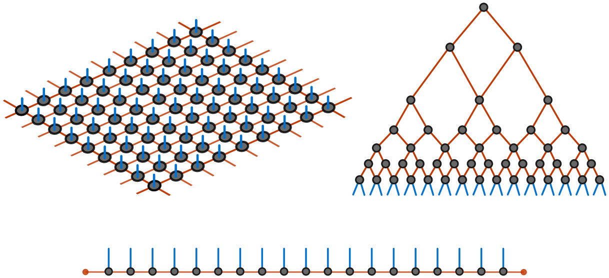
Tensor network states: a tool



Applications

- Quantum information theory
- Statistical Mechanics
- Quantum gravity
- **Many-body quantum**

Tensor network states: a tool



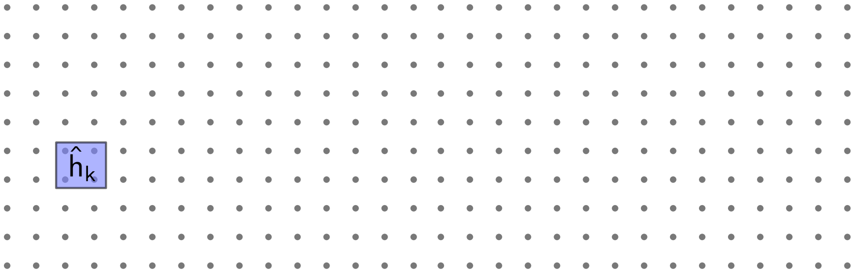
Applications

- Quantum information theory
- Statistical Mechanics
- Quantum gravity
- **Many-body quantum**

Negative theology

- **Not** covariant/geometric objects $g_{\mu\nu}$ or $R^{\sigma}_{\mu\nu\kappa}$
- **Not** tensor **models** [Rivasseau, Gurau, ...]

Many-body problem



Problem

Finding low energy states of

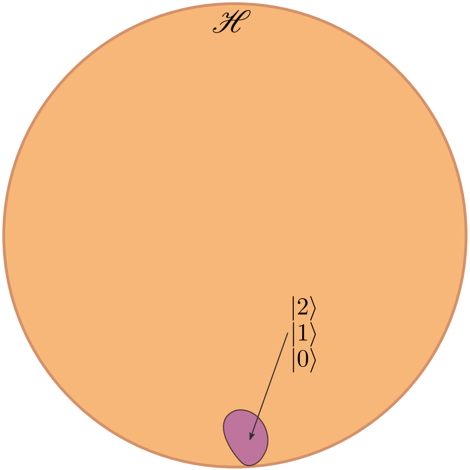
$$\hat{H} = \sum_{k=1}^{N} \hat{h}_k$$

is **hard** because $\dim \mathscr{H} \propto D^N$

Possible solutions

- Perturbation theory
- Monte Carlo
- Bootstrap IR fixed point
- **Variational optimization** (*e.g.* Mean Field, TCSA, tensor networks)

Variational optimization



Generic (spin $D/2$) state $\in \mathscr{H}$:

$$|\psi\rangle = \sum_{i_1, i_2, \cdots, i_n} c_{i_1, i_2, \cdots, i_N} |i_1, \cdots, i_N\rangle$$

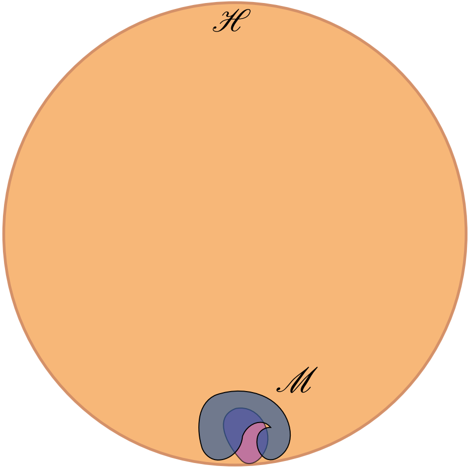
Exact variational optimization

To find the ground state:

$$|0\rangle = \min_{|\psi\rangle \in \mathscr{H}} \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

- $\dim \mathscr{H} = D^N$

Variational optimization



Generic (spin $D/2$) state $\in \mathscr{H}$:

$$|\psi\rangle = \sum_{i_1, i_2, \cdots, i_n} c_{i_1, i_2, \cdots, i_N} |i_1, \cdots, i_N\rangle$$

Approx. variational optimization

To find the ground state:

$$|0\rangle = \min_{|\psi\rangle \in \mathscr{M}} \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

- $\dim \mathscr{M} \propto \mathrm{Poly}(N)$ or fixed

An idea popular in many fields

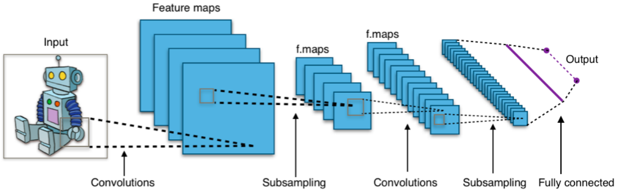
- **Mean field** approximation (of which TNS are an extension)

$$\psi(x_1, x_2, \cdots, x_n) = \psi_1(x_1)\,\psi_2(x_2)\cdots\psi_n(x_n)$$

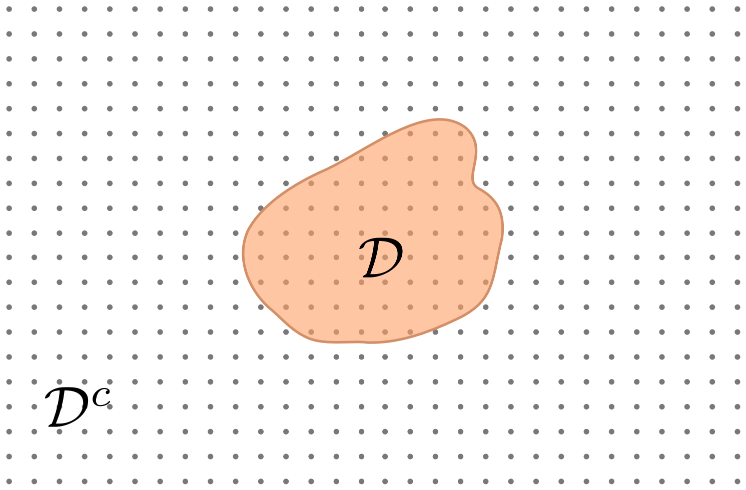
- Special variational wave functions in **Quantum chemistry** (whole industry of ansatz)
- **Moore-Read wavefunctions** in the study of the quantum Hall effect

$$\psi(x_1, x_2, \cdots, x_n) = \left\langle \hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n) \right\rangle_{\mathrm{CFT}}$$

- Fully connected and convolutional **neural networks** used in machine learning



Interesting states are weakly entangled



Low energy state

$|\psi\rangle = |0\rangle$ or $|1\rangle$ ...

Reduced density matrix

$\rho = \mathrm{tr}_{\mathcal{D}^c}\left[|\psi\rangle\langle\psi|\right]$

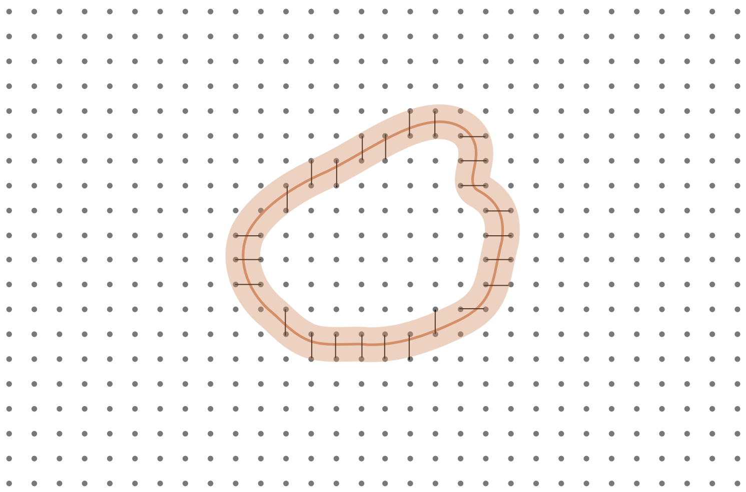
Entanglement entropy

$S = -\mathrm{tr}\left[\rho \log \rho\right]$

Area law

$$S \propto |\partial \mathcal{D}|$$

Interesting states are weakly entangled



Low energy state

$|\psi\rangle = |0\rangle$ or $|1\rangle$ ...

Reduced density matrix

$\rho = \mathrm{tr}_{\mathcal{D}^c}\left[|\psi\rangle\langle\psi|\right]$

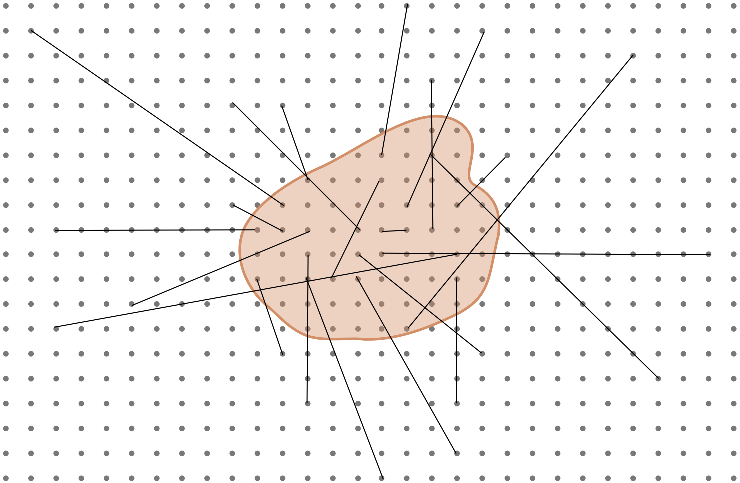
Entanglement entropy

$S = -\mathrm{tr}\left[\rho \log \rho\right]$

Area law

$$S \propto |\partial\mathcal{D}|$$

Typical states are strongly entangled



Random state

$|\psi\rangle = U_{\mathrm{Haar}}|\mathrm{trivial}\rangle$

Reduced density matrix

$\rho = \mathrm{tr}_{\mathcal{D}^c}\left[|\psi\rangle\langle\psi|\right]$

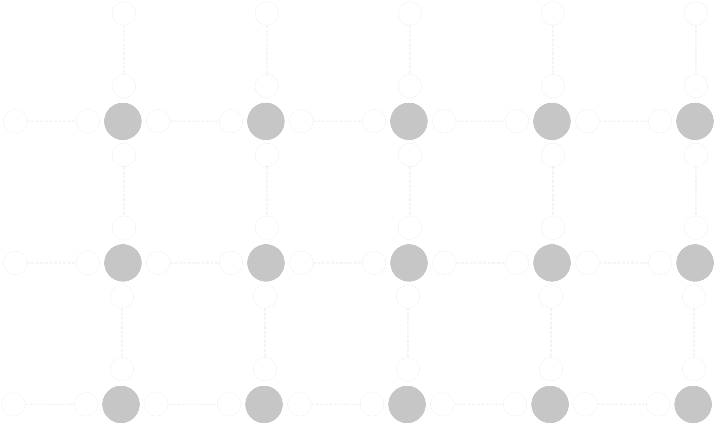
Entanglement entropy

$S = -\mathrm{tr}\left[\rho \log \rho\right]$

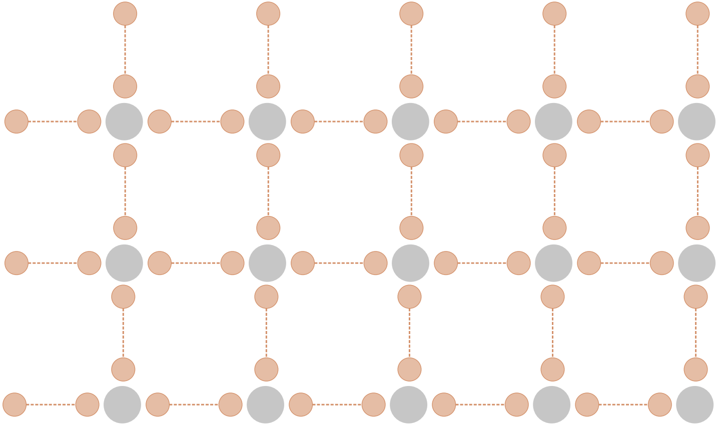
Volume law

$$S \propto |\mathcal{D}|$$

Constructing weakly entangled states



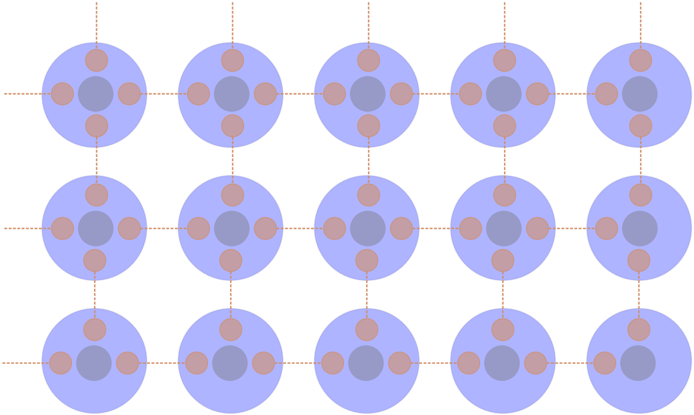
Constructing weakly entangled states



1. Put auxiliary **maximally entangled** states between sites

 $= \sum_{j=1}^{\chi} |j\rangle|j\rangle$

Constructing weakly entangled states



1. Put auxiliary **maximally entangled** states between sites

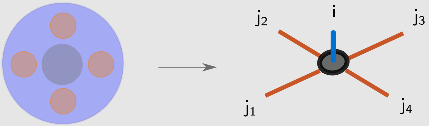
 $= \sum_{j=1}^{\chi} |j\rangle |j\rangle$

2. Map to initial Hilbert space on each site

 $= A : \mathbb{C}^{4\chi} \to \mathbb{C}^{D}$

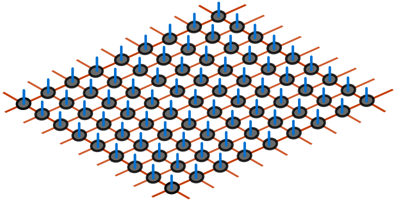
Tensor network states: definition

Why "tensor" network?



$$A : \mathbb{C}^{4\chi} \to \mathbb{C}^{d} \quad \longrightarrow \quad A^{i}_{j_1,j_2,j_3,j_4}$$

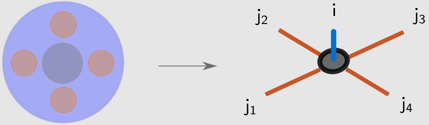
$|A\rangle =$



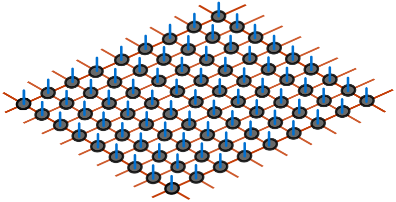
with tensor contractions on links

Tensor network states: definition

Why "tensor" network?



$$A : \mathbb{C}^{4\chi} \to \mathbb{C}^{d} \quad \longrightarrow \quad A^{i}_{j_1, j_2, j_3, j_4}$$

$|A\rangle =$ 

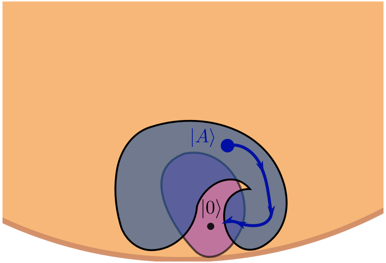
with tensor contractions on links

Optimization

Find best A for fixed χ $\quad (D \times \chi^4$ coeff.)

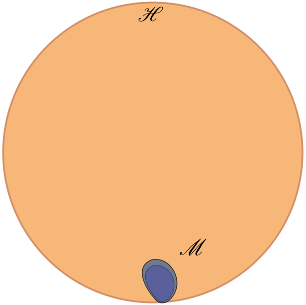
$$E_0 \simeq \min_A \frac{\langle A|\hat{H}|A\rangle}{\langle A|A\rangle}$$

for example go down $\frac{\partial E}{\partial A^{i}_{j_1, j_2, j_3, j_4}}$



Some facts

$d = 1$ spatial dimension

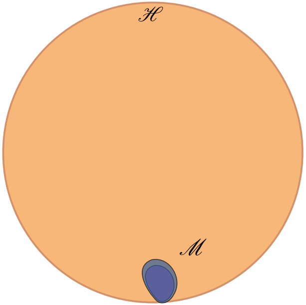


Theorems (colloquially)

1. For gapped H, tensor network states $|A\rangle$ approximate well $|0\rangle$ with χ fixed
2. **All** $|A\rangle$ are ground states of gapped H

Some facts

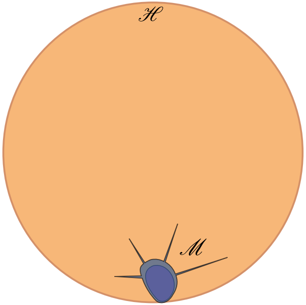
$d = 1$ spatial dimension



Theorems (colloquially)

1. For gapped H, tensor network states $|A\rangle$ approximate well $|0\rangle$ with χ fixed
2. **All** $|A\rangle$ are ground states of gapped H

$d \geqslant 2$ spatial dimension



Folklore

1. For gapped H, tensor network states $|A\rangle$ approximate well $|0\rangle$ with χ fixed
2. **Most** $|A\rangle$ are ground states of gapped H

Limitations

Hard to contract in $d \geqslant 2$

In $d \geqslant 2$ one can have:

- $|A\rangle$ known
- $\langle A|\hat{\mathcal{O}}_i\hat{\mathcal{O}}_j|A\rangle$ hard to compute exactly

Limitations

Hard to contract in $d \geqslant 2$

In $d \geqslant 2$ one can have:

- $|A\rangle$ known
- $\langle A|\hat{\mathcal{O}}_i\hat{\mathcal{O}}_j|A\rangle$ hard to compute exactly

Expressive but opaque

Generally hard to interpret

- Tensor carries IR-irrelevant information
- Hard to constrain long distance behavior

Limitations

Hard to contract in $d \geqslant 2$

In $d \geqslant 2$ one can have:

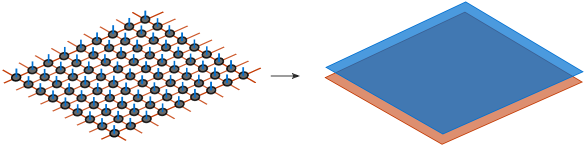
- $|A\rangle$ known
- $\langle A|\hat{\mathcal{O}}_i\hat{\mathcal{O}}_j|A\rangle$ hard to compute exactly

Expressive but opaque

Generally hard to interpret

- Tensor carries IR-irrelevant information
- Hard to constrain long distance behavior

$\Longrightarrow$ Go to the continuum and **QFT**: Major objective and challenge

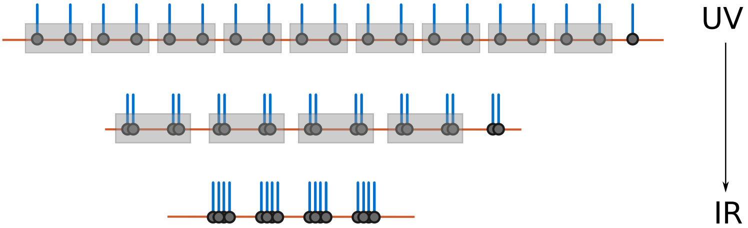


discrete tensor network

continuum description ??

Continuous Matrix Product states

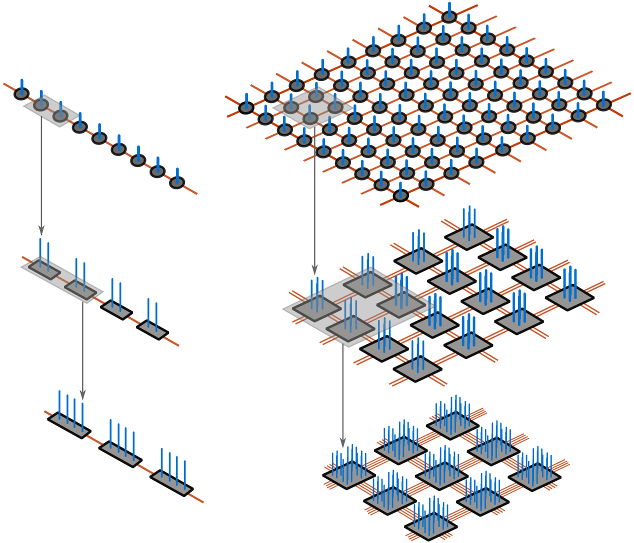
[Verstraete & Cirac 2010]: continuum limit of **M**atrix **P**roduct **S**tates ($d = 1$ tensor networks)



Works for Lieb-Liniger model (boson with contact interactions), ϕ^4, etc.
Best method on the market for $1+1$ QFT

But no version for $d+1$ QFT, even "no-go" theorems

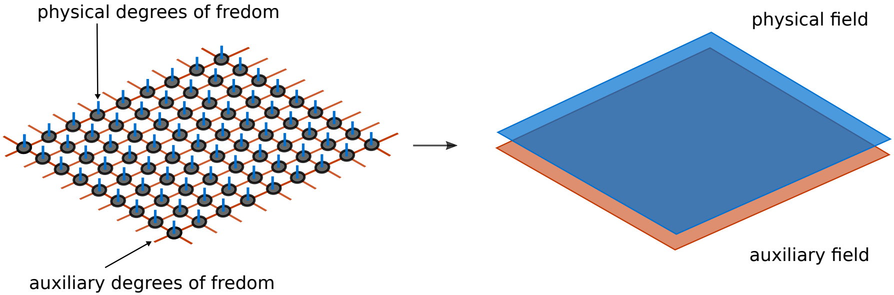
Continuous Tensor Networks: blocking



Upon **blocking**:

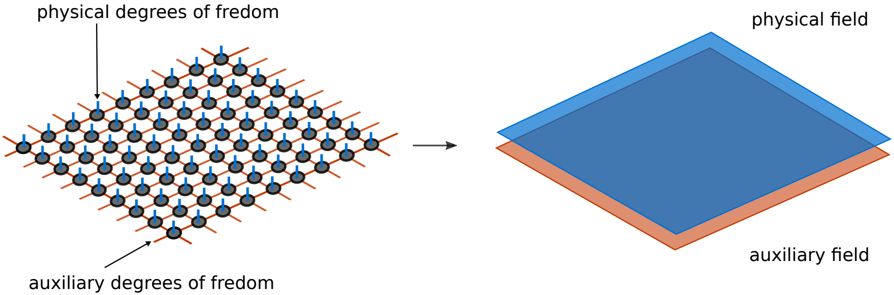
- ◇ The **physical** Hilbert space dimension D increases
- ◇ The **bond** (auxiliary space) dimension χ increases too

Result



AT, J. I. Cirac, *Phys. Rev. X 2019 (in print)*

Result



AT, J. I. Cirac, *Phys. Rev. X 2019 (in print)*

Continuous tensor network state (heuristically)

State $|\alpha\rangle$ of $d+1$ QFT from an auxiliary d dimensional theory of random fields ϕ:

$$|\alpha\rangle = \int \mathcal{D}\phi \, \exp\left\{ -\int \mathrm{d}^d x \, \mathcal{L}[\phi(x)] - \alpha[\phi(x)] \underbrace{\hat{\psi}^\dagger(x)}_{\text{creation}} \right\} |\Omega\rangle$$

1. Genuine continuum limit of discrete tensor networks
2. The toolbox is translated to the continuum

Future

Reopens the field after 8 years of only $d = 1$
So far, success **expected** from success in the discrete and continuous $d = 1$

Future

Reopens the field after 8 years of only $d = 1$
So far, success **expected** from success in the discrete and continuous $d = 1$

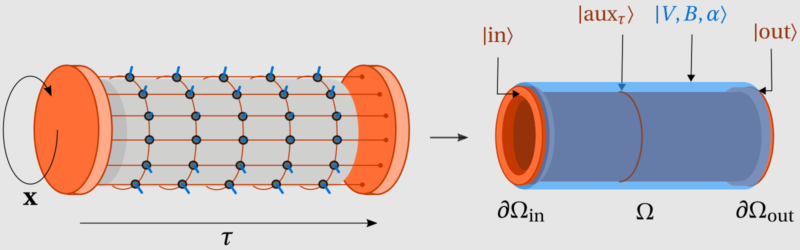
New **non-perturbative** method, how will it fare?

Future

Reopens the field after 8 years of only $d = 1$
So far, success **expected** from success in the discrete and continuous $d = 1$

New **non-perturbative** method, how will it fare?

Continuous tensor network states (cTNS) for dimensional reduction

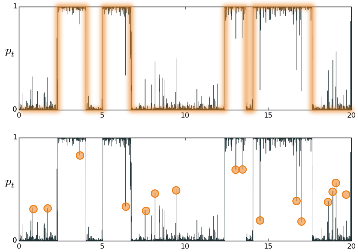


Contracting a cTNS in $2d$ = Solving χ field theories in $1d$ = Optimizing χ cTNS in $1d$

One can trade a dimension for a variational optimization

Summary: 2 fields, 2 main results

Continuous quantum measurement

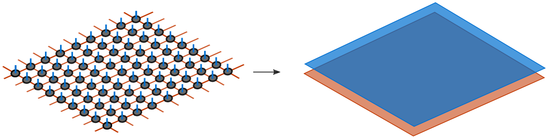


Mathematical understanding of stochastic dynamics to help control quantum systems in the lab

Main results:

- Quantum jumps
- Spikes

Tensor networks for QFT



Extend a powerful **variational method** from the lattice to the continuum

Main results:

- An ansatz of continuous tensor network state
- Promising non-perturbative methods for QFT

Bonus slides

–

Matrix product states

$$|\psi\rangle = \sum_{i_1, i_2, \cdots, i_n} c_{i_1, i_2, \cdots, i_n} |i_1, \cdots, i_n\rangle$$

Matrix product states

$$|\psi\rangle = \sum_{i_1, i_2, \cdots, i_n} c_{i_1, i_2, \cdots, i_n} |i_1, \cdots, i_n\rangle$$

Matrix Product States (MPS)

$$|A, L, R\rangle = \sum_{i_1, i_2, \cdots, i_n} \langle \mathrm{L}| A_{i_1}(1) A_{i_2}(2) \cdots A_{i_n}(n) |\mathrm{R}\rangle \, |i_1, \cdots, i_n\rangle$$

- A_i are $D \times D$ complex matrices
- A is a $2 \times D \times D$ tensor $[A_i]_{k,l}$
- $|L\rangle$ and $|R\rangle$ are D-vectors.

Matrix product states

$$|\psi\rangle = \sum_{i_1, i_2, \cdots, i_n} c_{i_1, i_2, \cdots, i_n} \, |i_1, \cdots, i_n\rangle$$

Matrix Product States (MPS)

$$|A, L, R\rangle = \sum_{i_1, i_2, \cdots, i_n} \langle \mathrm{L}|A_{i_1}(1) A_{i_2}(2) \cdots A_{i_n}(n)|\mathrm{R}\rangle \, |i_1, \cdots, i_n\rangle$$

- A_i are $D \times D$ complex matrices
- A is a $2 \times D \times D$ tensor $[A_i]_{k,l}$
- $|L\rangle$ and $|R\rangle$ are D-vectors.

Remark: actually equivalent with the density matrix renormalization group (DMRG)

Matrix product states

$$|\psi\rangle = \sum_{i_1, i_2, \cdots, i_n} c_{i_1, i_2, \cdots, i_n} |i_1, \cdots, i_n\rangle$$

Matrix Product States (MPS)

$$|A, L, R\rangle = \sum_{i_1, i_2, \cdots, i_n} \langle \mathrm{L}|A_{i_1}(1) A_{i_2}(2) \cdots A_{i_n}(n)|\mathrm{R}\rangle \, |i_1, \cdots, i_n\rangle$$

- A_i are $D \times D$ complex matrices
- A is a $2 \times D \times D$ tensor $[A_i]_{k,l}$
- $|L\rangle$ and $|R\rangle$ are D-vectors.

Remark: actually equivalent with the density matrix renormalization group (DMRG)

- ◊ $n \times 2 \times D^2$ parameters instead of 2^n
- ◊ D is the **bond dimension** and encodes the size of the variational class

Graphical notation

$|A, L, R\rangle = \sum_{i_1, i_2, \cdots, i_n} \langle \mathrm{L}| A_{i_1}(1) A_{i_2}(2) \cdots A_{i_n}(n) |\mathrm{R}\rangle \; |i_1, \cdots, i_n\rangle$

Notation: $[A_i]_{\mathbf{k},\mathbf{l}} =$  and k —— $l = \sum \delta_{k,l}$ gives:

$|A, L, R\rangle =$ 

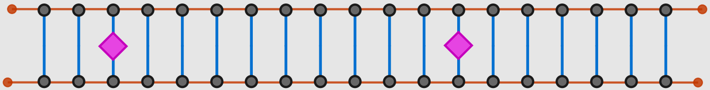
Graphical notation

$$|A, L, R\rangle = \sum_{i_1, i_2, \cdots, i_n} \langle \mathrm{L}| A_{i_1}(1) A_{i_2}(2) \cdots A_{i_n}(n) |\mathrm{R}\rangle \; |i_1, \cdots, i_n\rangle$$

Notation: $[A_i]_{\mathbf{k,l}} =$  and k ——— $l = \sum \delta_{k,l}$ gives:

$|A, L, R\rangle =$ 

Example: computation of correlations

$\langle A | \mathcal{O}(i_k) \mathcal{O}(i_\ell) | A \rangle =$ 

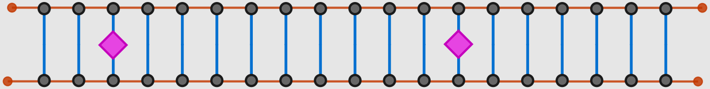
Graphical notation

$$|A, L, R\rangle = \sum_{i_1, i_2, \cdots, i_n} \langle \mathrm{L}|A_{i_1}(1) A_{i_2}(2) \cdots A_{i_n}(n)|\mathrm{R}\rangle \; |i_1, \cdots, i_n\rangle$$

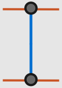
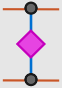
Notation: $[A_i]_{\mathbf{k},\mathbf{l}} =$  and k —— $l = \sum \delta_{k,l}$ gives:

$|A, L, R\rangle =$ 

Example: computation of correlations

$\langle A|\mathcal{O}(i_k)\mathcal{O}(i_\ell)|A\rangle =$ 

can be done efficiently by iterating 2 maps:

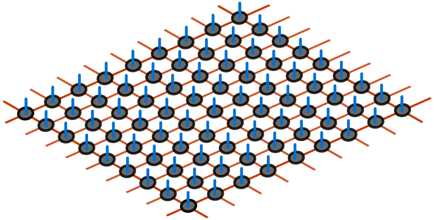
$\Phi =$  and $\Phi_{\mathcal{O}} =$ 

The contraction for a $d = 1$ system, can be seen as an open-system dynamics in $d = 0$.

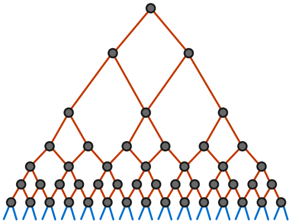
Generalizations: different tensor networks

Matrix Product States (MPS)

Projected Entangled Pair States (PEPS)



Multi-scale Entanglement Renormalization Ansatz (MERA)



Some facts

A list of theorems [very colloquially]:

- **Expressiveness** [trivial] Tensor Network States cover $\mathscr{H}$ when $D \propto 2^n$
- **Area law** The entanglement of a subregion of space scales as its area for a TNS
- **Efficiency** [gapped] Matrix Product States approximate well the ground states of gapped systems in 1 spatial dimension
- **Efficiency** [critical] Multi-scale Entanglement Renormalization Ansatz (MERA) approximate well the ground states of critical systems in 1 spatial dimension.
- **Symmetries** Physical symmetries can be implemented locally on the bond space
- **Inverse problem** TNS are the ground state of a local parent Hamiltonian

Successes and limits

Successes

- ♡ Arbitrary precision for $1d$ quantum systems
- ♡ Classification of topological phases in $1d$ and $2d$
- ♡ Progress on non-Abelian lattice Gauge theories
- ♡ AdS/CFT toy models

Limits

- ♠ Hard to contract in $d \geqslant 2$
- ♠ No continuum limit in $d \geqslant 2$
- ♠ Lack of analytic techniques

Successes and limits

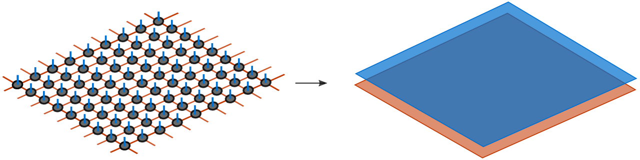
Successes

- ♡ Arbitrary precision for $1d$ quantum systems
- ♡ Classification of topological phases in $1d$ and $2d$
- ♡ Progress on non-Abelian lattice Gauge theories
- ♡ AdS/CFT toy models

Limits

- ♠ Hard to contract in $d \geqslant 2$
- ♠ No continuum limit in $d \geqslant 2$
- ♠ Lack of analytic techniques

Can one apply tensor network techniques directly in the continuum, to QFT?



Lots of “Continuous tensor network” concepts

Tensor networks for quantum states $|\psi\rangle$

- 
  MPS $\to$ cMPS
  [Verstraete & Cirac 2010]

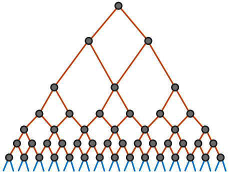
Lots of "Continuous tensor network" concepts

Tensor networks for quantum states $|\psi\rangle$

- 

  MPS $\rightarrow$ cMPS

  [Verstraete & Cirac 2010]

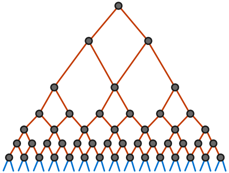
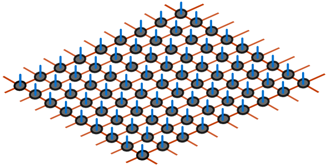
- 

  MERA $\longrightarrow$ cMERA

  [Haegeman et al. 2013]

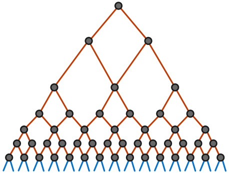
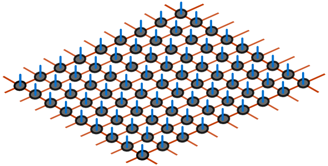
Lots of “Continuous tensor network” concepts

Tensor networks for quantum states $|\psi\rangle$

- 
  MPS $\rightarrow$ cMPS
  [Verstraete & Cirac 2010]
- 
  MERA $\longrightarrow$ cMERA
  [Haegeman et al. 2013]
- 
  PEPS $\rightarrow$ cPEPS

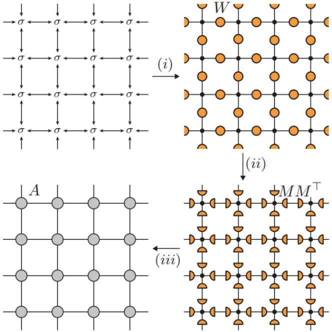
Lots of "Continuous tensor network" concepts

Tensor networks for quantum states $|\psi\rangle$

- 
  MPS $\rightarrow$ cMPS
  [Verstraete & Cirac 2010]
- 
  MERA $\longrightarrow$ cMERA
  [Haegeman et al. 2013]
- 
  PEPS $\rightarrow$ cPEPS

Tensor networks for partition functions $Z(\beta)$

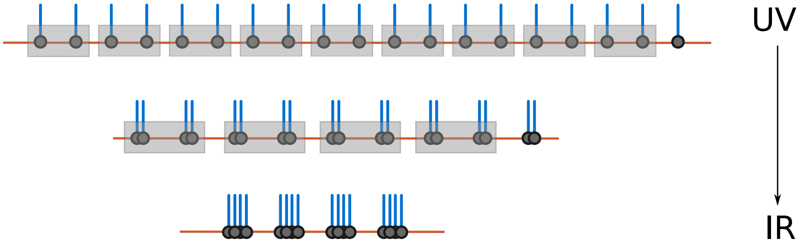
- StatMech in d
- Euclidean quantum in $d+1$



[Qi Hu et al. 2018]

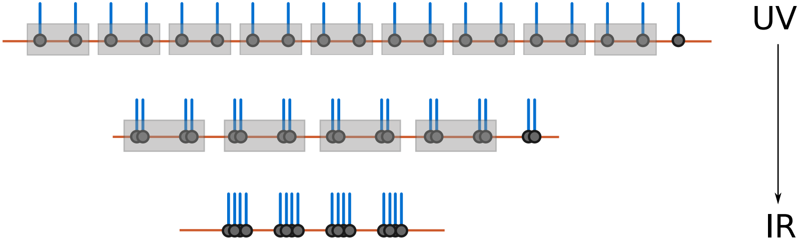
Continuous Matrix Product States (cMPS)

Taking the continuum limit of a MPS



Continuous Matrix Product States (cMPS)

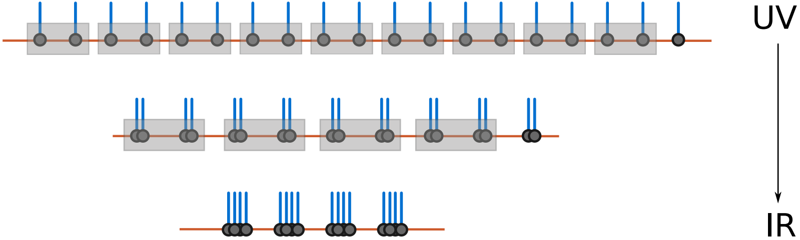
Taking the continuum limit of a MPS



- the bond dimension D stays fixed

Continuous Matrix Product States (cMPS)

Taking the continuum limit of a MPS



- the bond dimension D stays fixed
- the local physical dimension explodes $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \longrightarrow \mathscr{F}(L^2([x, x+dx]))$.
  $\Longrightarrow$ **Spins** become **fields** – ($\simeq$ central limit theorem $\simeq$)

Continuous Matrix Product States

Type of ansatz for bosons on a fine grained $d = 1$ lattice

- Matrices $A_{i_k}(x)$ where the index i_k corresponds to $\psi^{\dagger i_k}(x)|0\rangle$ in physical space.

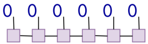
Informal cMPS definition

$$A_0 = \mathbb{1} + \varepsilon Q$$
$$A_1 = \varepsilon R$$
$$A_2 = \frac{(\varepsilon R)^2}{\sqrt{2}}$$
$$\dots$$
$$A_n = \frac{(\varepsilon R)^n}{\sqrt{n}}$$
$$\dots$$

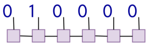
so we go from ∞ to 2 matrices

Fixed by:

- Finite particle number



$\propto \quad 1$



$\propto \quad \varepsilon$

- Consistency

Continuous Matrix Product States

Definition

$$|Q, R, \omega\rangle = \langle \omega_L | \mathcal{P} \exp \left\{ \int_0^L dx \; Q \otimes \mathbb{1} + R \otimes \psi^\dagger(x) \right\} |\omega_R\rangle \; |0\rangle$$

- Q, R are $D \times D$ matrices,
- $|\omega_L\rangle$ and $|\omega_R\rangle$ are boundary vectors $\in \mathbb{C}^D$, for p.b.c. $\langle \omega_L | \cdot | \omega_R \rangle \to \mathrm{tr}[\,\cdot\,]$
- $[\psi(x), \psi^\dagger(y)] = \delta(x - y)$

Idea:

Continuous Matrix Product States

Definition

$$|Q, R, \omega\rangle = \langle \omega_L | \mathcal{P} \exp \left\{ \int_0^L dx \; Q \otimes \mathbb{1} + R \otimes \psi^\dagger(x) \right\} |\omega_R\rangle \; |0\rangle$$

- Q, R are $D \times D$ matrices,
- $|\omega_L\rangle$ and $|\omega_R\rangle$ are boundary vectors $\in \mathbb{C}^D$, for p.b.c. $\langle \omega_L | \cdot | \omega_R \rangle \to \mathrm{tr}[\,\cdot\,]$
- $[\psi(x), \psi^\dagger(y)] = \delta(x - y)$

Idea:

$$\begin{aligned} A(x) &\simeq A_0 \mathbb{1} + A_1 \psi^\dagger(x) \\ &\simeq \mathbb{1} \otimes \mathbb{1} + \varepsilon Q \otimes \mathbb{1} + \varepsilon R \otimes \psi^\dagger(x) \\ &\simeq \exp \left[\varepsilon \left(Q \otimes \mathbb{1} + R \otimes \psi^\dagger(x) \right) \right] \end{aligned}$$

Computations

Some correlation functions

$$\left\langle \hat{\psi}(x)^\dagger \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{TL}(R \otimes \overline{R})\right]$$

$$\left\langle \hat{\psi}(x)^\dagger \hat{\psi}(0)^\dagger \hat{\psi}(0) \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{T(L-x)}(R \otimes \overline{R}) e^{Tx} (R \otimes \overline{R})\right]$$

$$\left\langle \hat{\psi}(x)^\dagger \left[-\frac{d^2}{dx^2} \right] \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{TL}([Q, R] \otimes [\overline{Q}, \overline{R}])\right]$$

with $T = Q \otimes \mathbb{1} + \mathbb{1} \otimes \bar{Q} + R \otimes \bar{R}$

Example

Lieb-Liniger Hamiltonian

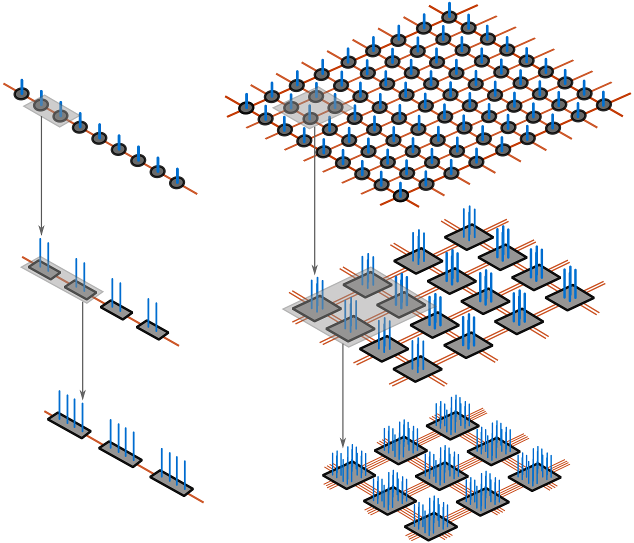
$$\mathcal{H} = \int_{-\infty}^{+\infty} dx \left[\frac{d\hat{\psi}^\dagger(x)}{dx} \frac{d\hat{\psi}(x)}{dx} + c \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(x) \hat{\psi}(x) \hat{\psi}(x) \right]$$

Solve by **minimizing**:

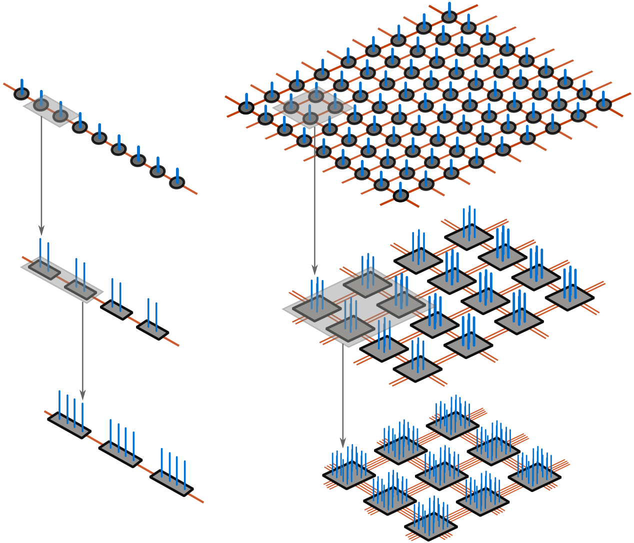
$$\langle Q, R | \mathcal{H} | Q, R \rangle = f(Q, R)$$

with fixed particle density $\langle Q, R | \psi^\dagger(x) \psi(x) | Q, R \rangle$.

Continuous Tensor Networks: blocking



Continuous Tensor Networks: blocking



Upon blocking:

- The **physical** Hilbert space dimension d increases (idem cMPS $\Longrightarrow$ physical field)
- The **bond** dimension D increases too

Choice of trivial tensor

For **MPS**, not much choice:

 $=$ $+ \quad \varepsilon \cdots$

$$= \mathbb{1} \otimes |0\rangle + \varepsilon\, Q \otimes |0\rangle + \varepsilon\, R \otimes \psi^{\dagger}(x)|0\rangle$$

Choice of trivial tensor

For **MPS**, not much choice:

 $=$ $+ \; \varepsilon \cdots$

$$= \mathbb{1} \otimes |0\rangle + \varepsilon\, Q \otimes |0\rangle + \varepsilon\, R \otimes \psi^\dagger(x)|0\rangle$$

For **TNS** in $d \geqslant 2$, many options:

1. Take a δ between all legs $\sim$ GHZ state $T^{(0)} =$ 
   $\Longrightarrow$ trivial geometry

Choice of trivial tensor

For **MPS**, not much choice:

 $=$ $+ \;\; \varepsilon \cdots$

$$= \mathbb{1} \otimes |0\rangle + \varepsilon\, Q \otimes |0\rangle + \varepsilon\, R \otimes \psi^\dagger(x)|0\rangle$$

For **TNS** in $d \geqslant 2$, many options:

1. Take a δ between all legs $\sim$ GHZ state $T^{(0)} =$ 
   $\Longrightarrow$ trivial geometry
2. Take two identities $T^{(0)} =$ 
   $\Longrightarrow$ breakdown of Euclidean invariance

Choice of trivial tensor

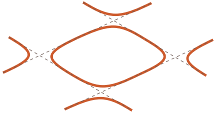
For **MPS**, not much choice:

 $=$ $+ \ \varepsilon \cdots$

$$= \mathbb{1} \otimes |0\rangle + \varepsilon\, Q \otimes |0\rangle + \varepsilon\, R \otimes \psi^\dagger(x)|0\rangle$$

For **TNS** in $d \geqslant 2$, many options:

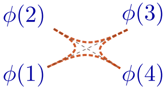
1. Take a δ between all legs ~ GHZ state $T^{(0)} =$ 
   $\Longrightarrow$ trivial geometry
2. Take two identities $T^{(0)} =$ 
   $\Longrightarrow$ breakdown of Euclidean invariance
3. Take the sum of pairs of identities in both directions $T^{(0)} =$  $+$ 



Ansatz

1 – Take a "Trivial" tensor:

$$T^{(0)}_{\phi(1),\phi(2),\phi(3),\phi(4)} =$$



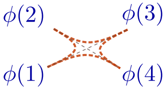
$$\sim \exp\left\{ \frac{-1}{2} \sum_{k=1}^{D} [\phi_k(1) - \phi_k(2)]^2 + [\phi_k(2) - \phi_k(3)]^2 + [\phi_k(3) - \phi_k(4)]^2 + [\phi_k(4) - \phi_k(1)]^2 \right\}$$

The indices ϕ are in $\mathbb{R}^D$ (and **not** $1, \cdots, D$)

Ansatz

1 – Take a "Trivial" tensor:

$$T^{(0)}_{\phi(1),\phi(2),\phi(3),\phi(4)} =$$



$$\sim \exp\left\{ \frac{-1}{2} \sum_{k=1}^{D} [\phi_k(1) - \phi_k(2)]^2 + [\phi_k(2) - \phi_k(3)]^2 + [\phi_k(3) - \phi_k(4)]^2 + [\phi_k(4) - \phi_k(1)]^2 \right\}$$

The indices ϕ are in $\mathbb{R}^D$ (and **not** $1, \cdots, D$)

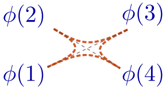
2 – And add a "correction":

$$\exp\left\{ -\varepsilon^2 V\left[\phi(1), \cdots, \phi(4)\right] + \varepsilon^2 \alpha \left[\phi(1), \cdots, \phi(4)\right] \psi^\dagger(x) \right\}$$

Ansatz

1 – Take a "Trivial" tensor:

$$T^{(0)}_{\phi(1),\phi(2),\phi(3),\phi(4)} =$$



$$\sim \exp\left\{ \frac{-1}{2} \sum_{k=1}^{D} [\phi_k(1) - \phi_k(2)]^2 + [\phi_k(2) - \phi_k(3)]^2 + [\phi_k(3) - \phi_k(4)]^2 + [\phi_k(4) - \phi_k(1)]^2 \right\}$$

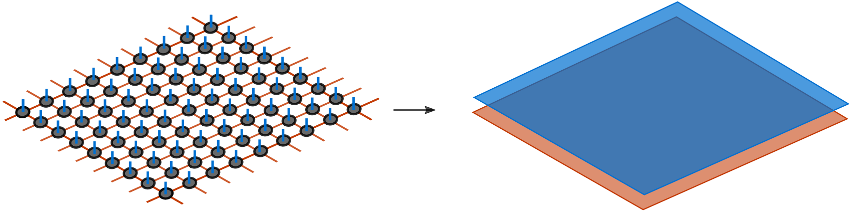
The indices ϕ are in $\mathbb{R}^D$ (and **not** $1, \cdots, D$)

2 – And add a "correction":

$$\exp\left\{ -\varepsilon^2 V[\phi(1), \cdots, \phi(4)] + \varepsilon^2 \alpha [\phi(1), \cdots, \phi(4)] \psi^\dagger(x) \right\}$$

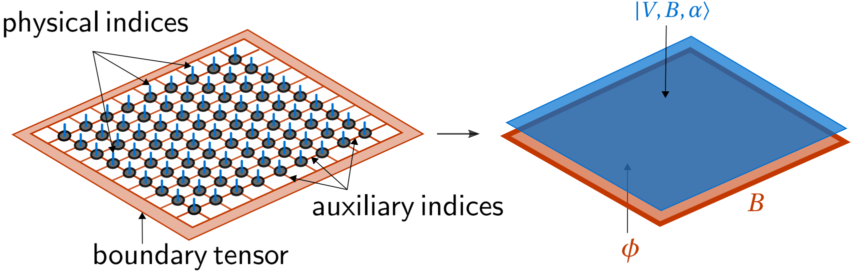
3 – Realize tensor contraction = functional integral and trivial tensor gives free field measure.

Functional integral definition



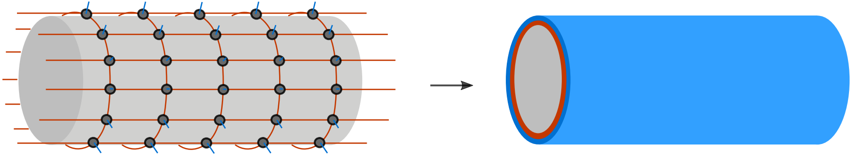
$$|V,\alpha\rangle = \int \mathcal{D}\phi \, \exp\left\{ -\int_\Omega \mathrm{d}^d x \, \frac{1}{2} \sum_{k=1}^{D} [\nabla \phi_k(x)]^2 + V[\phi(x)] - \alpha[\phi(x)] \, \hat{\psi}^\dagger(x) \right\} |0\rangle$$

Functional integral definition



$$|V, B, \alpha\rangle = \int \mathcal{D}\phi \, B(\phi|_{\partial\Omega}) \exp\left\{ -\int_{\Omega} \mathrm{d}^d x \, \frac{1}{2} \sum_{k=1}^{D} [\nabla \phi_k(x)]^2 + V[\phi(x)] - \alpha[\phi(x)] \, \psi^\dagger(x) \right\} |0\rangle$$

Operator definition



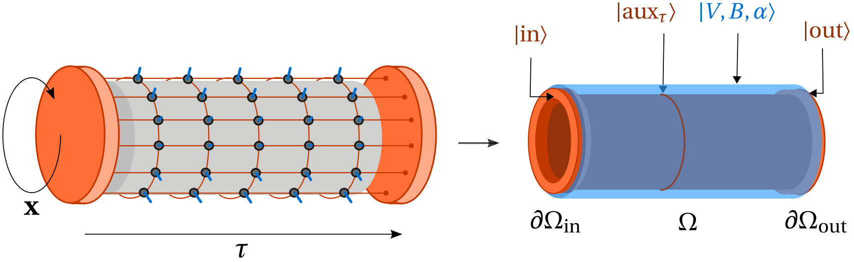
$$|V,\alpha\rangle =$$

$$\mathrm{tr}\left[\mathcal{T}\exp\left(-\int_0^T d\tau \int_S dx\, \frac{\hat{\pi}_k(x)\hat{\pi}_k(x)}{2} + \frac{\nabla\hat{\phi}_k(x)\nabla\hat{\phi}_k(x)}{2} + V[\hat{\phi}(x)] - \alpha[\hat{\phi}(x)]\,\psi^\dagger(\tau,x)\right)\right]|0\rangle$$

where:

- $\hat{\phi}_k(x)$ and $\hat{\pi}_k(x)$ are k independent canonically conjugated pairs of (auxiliary) field *operators*: $[\hat{\phi}_k(x),\hat{\phi}_l(y)] = 0$, $[\hat{\pi}(x)_k,\hat{\pi}_l(y)] = 0$, and $[\hat{\phi}_k(x),\hat{\pi}_l(y)] = i\delta_{k,l}\,\delta(x-y)$ acting on a space of $d-1$ dimensions.

Operator definition



$$|V, B, \alpha\rangle =$$

$$\mathrm{tr}\left[\hat{B}\,\mathcal{T}\exp\left(-\int_0^T d\tau \int_S dx\, \frac{\hat{\pi}_k(x)\hat{\pi}_k(x)}{2} + \frac{\nabla\hat{\phi}_k(x)\nabla\hat{\phi}_k(x)}{2} + V[\hat{\phi}(x)] - \alpha[\hat{\phi}(x)]\,\psi^\dagger(\tau, x)\right)\right]|0\rangle$$

where:

- $\hat{\phi}_k(x)$ and $\hat{\pi}_k(x)$ are k independent canonically conjugated pairs of (auxiliary) field *operators*: $[\hat{\phi}_k(x), \hat{\phi}_l(y)] = 0$, $[\hat{\pi}(x)_k, \hat{\pi}_l(y)] = 0$, and $[\hat{\phi}_k(x), \hat{\pi}_l(y)] = i\delta_{k,l}\,\delta(x-y)$ acting on a space of $d-1$ dimensions.

Wave-function definition

A generic state $|\Psi\rangle$ in Fock space can be written:

$$|\Psi\rangle = \sum_{n=0}^{+\infty} \int_{\Omega^n} \frac{\varphi_n(x_1, \cdots, x_n)}{n!} \psi^\dagger(x_1) \cdots \psi^\dagger(x_n) |0\rangle$$

where ϕ_n is a symmetric n-particle wave-function

Functional integral representation

$$\varphi_n(x_1, \cdots, x_n) = \langle\, \alpha[\phi(x_1)] \cdots \alpha[\phi(x_n)]\, \rangle_{\text{aux}}$$

with:

$$\langle \cdot \rangle_{\text{aux}} = \int \mathcal{D}\phi \;\cdot\; B(\phi|_{\partial\Omega}) \exp\left[-\frac{1}{2}\int_\Omega \mathrm{d}^d x \, [\nabla\phi_k(x)]^2 + V[\phi(x)]\right]$$

- $\sim$ Moore-Read wave-function for Quantum Hall, but generic QFT

Expressivity and stability

How big are cTNS?

Stability

The sum of two cTNS of bond field dimension D_1 and D_2 is a cTNS with bond field dimension $D \leqslant D_1 + D_2 + 1$:

$$|V_1, \alpha_1\rangle + |V_2, \alpha_2\rangle = |W, \beta\rangle$$

Expressiveness

All states in the Fock space can be approximated by cTNS:

- A field coherent state is a cTNS with $D = 0$
- Stability allows to get all sums of field coherent states

Note: expressiveness can also be obtained with $D = 1$ but it is less natural. Flexibility in D makes the expressivity higher for restricted classes of V and α.

Computations

Define generating functional for normal ordered correlation functions

$$\mathcal{Z}_{j',j} = \frac{1}{\langle V,\alpha|V,\alpha\rangle} \langle V,\alpha| \exp\left(\int \mathrm{d}x\, j'(x)\psi^\dagger(x)\right) \exp\left(\int \mathrm{d}x\, j(x)\psi(x)\right) |V,\alpha\rangle$$

Operator representation

$$\mathcal{Z}_{j'j} = \mathrm{tr}\left[B \otimes B^* \mathcal{T} \exp\left\{\int_{-T/2}^{T/2} \left(\mathrm{T}_{j'j} - \int_S j \cdot j'\right)\right\}\right]$$

with **transfer matrix**:

$$\mathrm{T}_{j'j} = \int_S \mathrm{d}x\, \mathcal{H}(x) \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{H}^*(x) + \left(\alpha[\hat{\phi}(x)] + j'(x)\right) \otimes \left(\alpha[\hat{\phi}(x)]^* + j(x)\right)$$

and

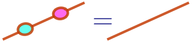
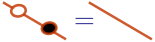
$$\mathcal{H}(x) = \sum_{k=1}^{D} \frac{[\hat{\pi}_k(\mathbf{x})]^2 + \left[\nabla\hat{\phi}_k(\mathbf{x})\right]^2}{2} + V[\hat{\phi}(x)]$$

$\Longrightarrow$ cMPS brought us from 1 to 0, cTNS bring us from d to $d-1$.

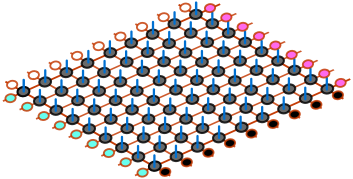
Redundancies

Discrete redundancy

Different elementary tensors are **equivalent**, they give the same state:

when  and 

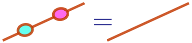
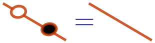
up to **boundary** terms:



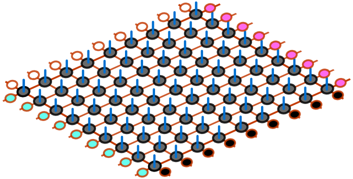
Redundancies

Discrete redundancy

Different elementary tensors are **equivalent**, they give the same state:

when  and 

up to **boundary** terms:

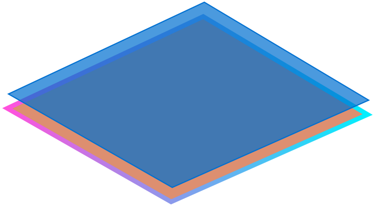


Continuum redundancy

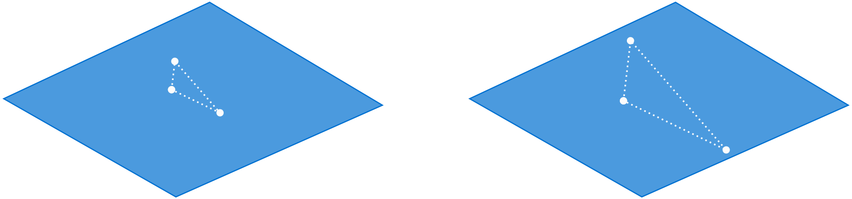
$$V(\phi) \to V(\phi) + \nabla \cdot \mathscr{F}[x, \phi(x)]$$

Just Stokes' theorem. If Ω has a boundary $\partial\Omega$:

$$\mathcal{D}[\phi] \to \mathcal{D}[\phi] \exp\left\{ \oint_{\partial\Omega} \mathrm{d}^{d-1}x \, \mathscr{F}[x, \phi(x)] \cdot \mathbf{n}(x) \right\}$$



Rescaling

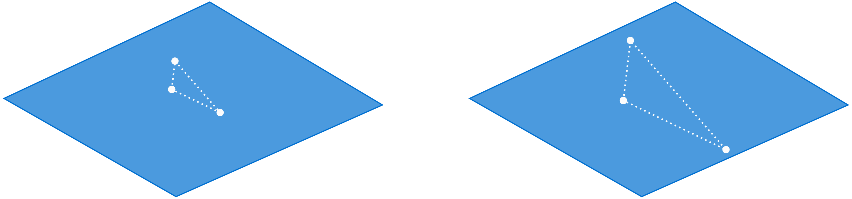


$$C(x_1, \cdots, x_n) = \langle T(1)|\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)|T(1)\rangle,$$

the objective is to find a tensor $T(\lambda)$ of new parameters such that:

$$C(\lambda x_1, \cdots, \lambda x_n) \propto \langle T(\lambda)|\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)|T(\lambda)\rangle.$$

Rescaling



$$C(x_1, \cdots, x_n) = \langle T(1) | \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) | T(1) \rangle,$$

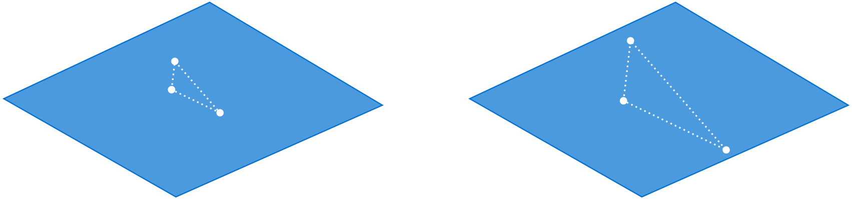
the objective is to find a tensor $T(\lambda)$ of new parameters such that:

$$C(\lambda x_1, \cdots, \lambda x_n) \propto \langle T(\lambda) | \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) | T(\lambda) \rangle.$$

Doable exactly:

$$V \to \lambda^d V \circ \lambda^{\frac{2-d}{2}} \quad \text{and} \quad \alpha \to \lambda^{\frac{d}{2}} \alpha \circ \lambda^{\frac{2-d}{2}}$$

Rescaling



$$C(x_1,\cdots,x_n) = \langle T(1)|\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)|T(1)\rangle,$$

the objective is to find a tensor $T(\lambda)$ of new parameters such that:

$$C(\lambda x_1,\cdots,\lambda x_n) \propto \langle T(\lambda)|\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)|T(\lambda)\rangle.$$

Doable exactly:

$$V \to \lambda^d V \circ \lambda^{\frac{2-d}{2}} \quad \text{and} \quad \alpha \to \lambda^{\frac{d}{2}} \alpha \circ \lambda^{\frac{2-d}{2}}$$

- $d = 2$, All powers of the field in V and α yield relevant couplings
- $d = 3$, The powers $p = 1, 2, 3, 4, 5$ of the field in V yield relevant $\Delta > 0$ couplings. $p = 6$ is marginal in V. For α, $p = 1, 2$ are relevant and $p = 3$ is marginal. All other p are irrelevant.

Renormalization

Scaling

- $d = 2$, All powers of the field in V and α yield relevant couplings
- $d = 3$, The powers $p = 1, 2, 3, 4, 5$ of the field in V yield relevant $\Delta > 0$ couplings. $p = 6$ is marginal in V. For α, $p = 1, 2$ are relevant and $p = 3$ is marginal. All other p are irrelevant.

For finite bond field dimension in $d = 3$, finite number of parameters for **renormalized** cTNS:

$$V(\phi) = A\phi + B\phi\phi + C\phi\phi\phi + D\phi\phi\phi\phi + E\phi\phi\phi\phi\phi + F\phi\phi\phi\phi\phi\phi$$
$$\alpha(\phi) = X\phi + Y\phi\phi + Z\phi\phi\phi$$

Proper renormalization procedure not checked yet

Getting back cMPS

One can get back cMPS with finite bond dimension by:

1. **Compactification** Take $d-1$ dimensions out of d to be very small

$$|V,B,\alpha\rangle \simeq \mathrm{tr}\left[\hat{B}\,\mathcal{T}\exp\left(-\int_0^T \mathrm{d}\tau \sum_{k=1}^{D} \frac{\hat{P}_k^2}{2} + V[\hat{X}] - \alpha[\hat{X}]\,\psi^\dagger(\tau)\right)\right]|0\rangle$$

$\implies$ Hilbert space of a quantum particle in D space dimensions.

2. **Quantization** Take V with D deep minima to force the auxiliary field to take only D possibilities

Generalization

For a general Riemanian manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, define:

$$|V, B, \alpha\rangle = \int \mathcal{D}\phi \, B(\phi|_{\partial\mathcal{M}}) \exp\left\{-\int_{\mathcal{M}} \mathrm{d}^d x \sqrt{g} \left(\frac{g^{\mu\nu}\partial_\mu\phi_k\partial_\nu\phi_k}{2} + V[\phi, \nabla\phi] - \alpha[\phi, \nabla\phi]\,\psi^\dagger\right)\right\}|0\rangle$$

i.e. add curvature and possible anisotropies in V and α

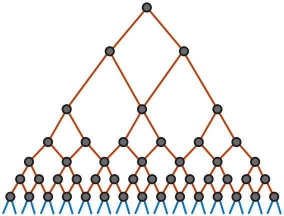
Generalization

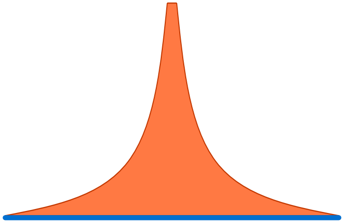
For a general Riemanian manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, define:

$$|V, B, \alpha\rangle = \int \mathcal{D}\phi \, B(\phi|_{\partial\mathcal{M}}) \exp\left\{-\int_{\mathcal{M}} \mathrm{d}^d x \sqrt{g} \left(\frac{g^{\mu\nu}\partial_\mu\phi_k\partial_\nu\phi_k}{2} + V[\phi, \nabla\phi] - \alpha[\phi, \nabla\phi]\,\psi^\dagger\right)\right\}|0\rangle$$

i.e. add curvature and possible anisotropies in V and α

Example: $\alpha[x, \phi, \nabla\phi]$ localized on the boundary and hyberbolic metrix g:





$\rightarrow$ **cMERA**-like in $d-1$ dimensions

Future

Limitations and work for the future

- Quite formal out of the Gaussian regime
- Computation through dimensional reduction not trivial
- Limited to bosonic field theories (so far)
- Gauge invariant states
- Can one say anything about topology?

Summary

$$|V, B, \alpha\rangle = \int \mathcal{D}\phi \, B(\phi|_{\partial\Omega}) \exp\left\{-\int_\Omega \mathrm{d}^d x \, \frac{1}{2} \sum_{k=1}^{D} [\nabla\phi_k(x)]^2 + V[\phi(x)] - \alpha[\phi(x)] \, \psi^\dagger(x)\right\} |0\rangle$$

Continuous tensor network states are natural continuum limits of tensor network states and natural higher d extensions of continuous matrix product states.

1. Obtained from discrete tensor networks
2. Can be made Euclidean invariant
3. **Motto of tensor networks**: trade a dimension for a variational optimization
4. Still need to be properly renormalized (in perturbative and RG sense)
5. Still needs to be used to approximate non-trivial non-Gaussian ground states