Tensor network states

from the discrete to the continuum

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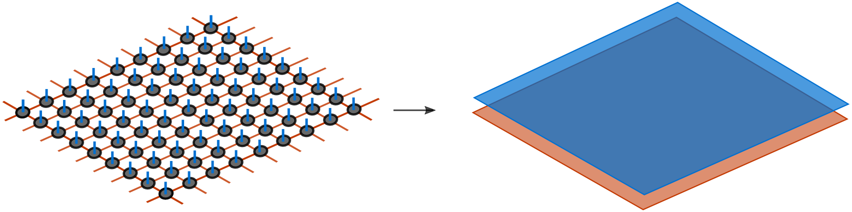

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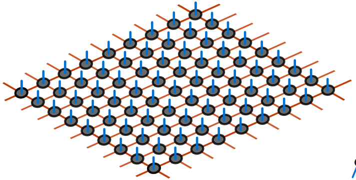
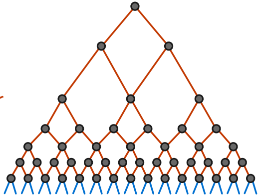
Tensor network states: a tool

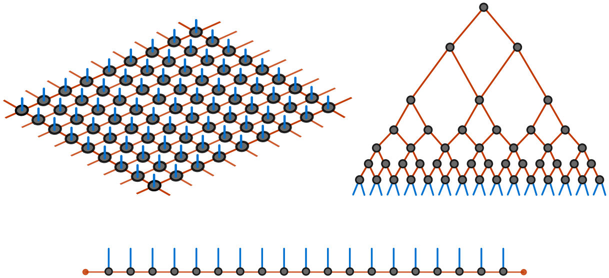




Applications

- Quantum information theory
- Statistical Mechanics
- Quantum gravity
- **Many-body quantum**

Tensor network states: a tool



Applications

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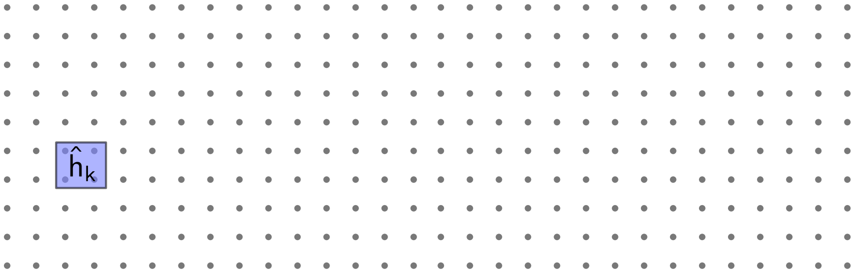
- **Many-body quantum**

Negative theology

- **Not** covariant/geometric objects $g_{\mu\nu}$ or $R^{\sigma}_{\mu\nu\kappa}$
- **Not** tensor **models** [Rivasseau, Gurau, ...]

Many-body problem



Problem

Finding low energy states of

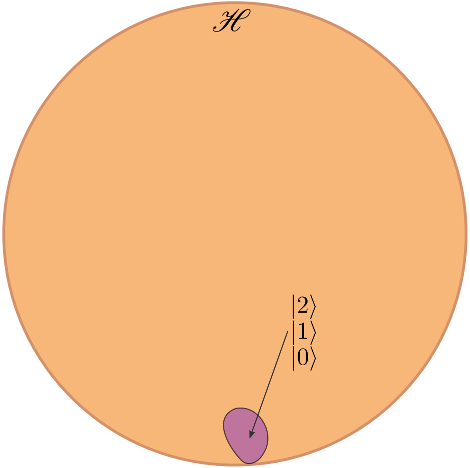
$$\hat{H} = \sum_{k=1}^{N} \hat{h}_k$$

is **hard** because $\dim \mathscr{H} \propto D^N$

Possible solutions

- Perturbation theory
- Monte Carlo
- Bootstrap IR fixed point
- **Variational optimization** (*e.g.* Mean Field, TCSA, tensor networks)

Variational optimization



Generic (spin $D/2$) state $\in \mathscr{H}$:

$$|\psi\rangle = \sum_{i_1, i_2, \cdots, i_n} c_{i_1, i_2, \cdots, i_N} |i_1, \cdots, i_N\rangle$$

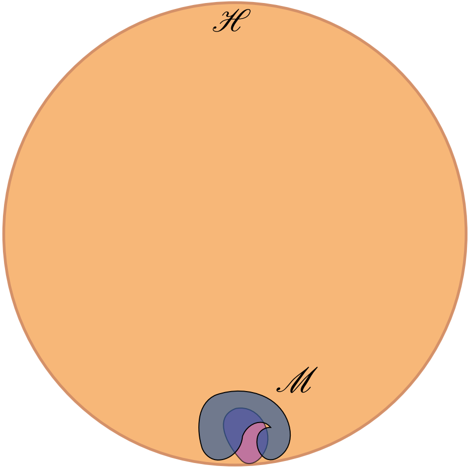
Exact variational optimization

To find the ground state:

$$|0\rangle = \min_{|\psi\rangle \in \mathscr{H}} \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

- $\dim \mathscr{H} = D^N$

Variational optimization



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Approx. variational optimization

To find the ground state:

$$|0\rangle = \min_{|\psi\rangle \in \mathscr{M}} \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

- $\dim \mathscr{M} \propto \mathrm{Poly}(N)$ or fixed

An idea popular in many fields

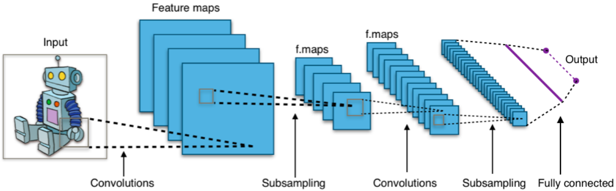
- **Mean field** approximation (of which TNS are an extension)

$$\psi(x_1, x_2, \cdots, x_n) = \psi_1(x_1)\,\psi_2(x_2)\cdots\psi_n(x_n)$$

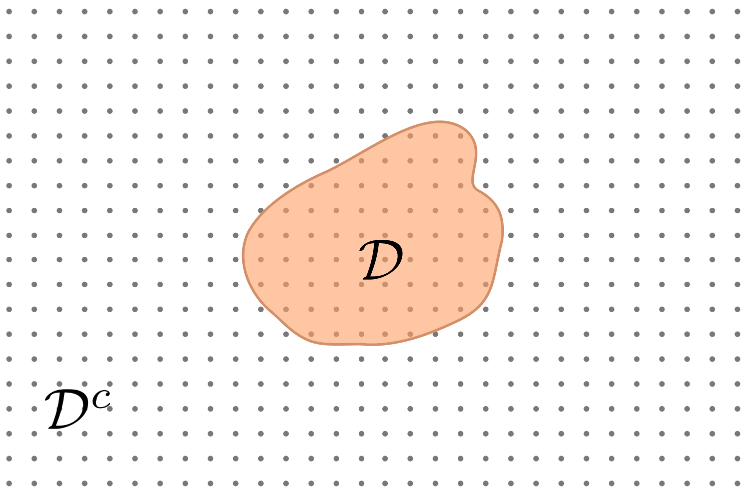
- Special variational wave functions in **Quantum chemistry** (whole industry of ansatz)
- **Moore-Read wavefunctions** in the study of the quantum Hall effect

$$\psi(x_1, x_2, \cdots, x_n) = \left\langle \hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n) \right\rangle_{\mathrm{CFT}}$$

- Fully connected and convolutional **neural networks** used in machine learning



Interesting states are weakly entangled



Low energy state

$|\psi\rangle = |0\rangle$ or $|1\rangle$ ...

Reduced density matrix

$\rho = \mathrm{tr}_{\mathcal{D}^c}\left[|\psi\rangle\langle\psi|\right]$

Entanglement entropy

$S = -\mathrm{tr}\left[\rho \log \rho\right]$

Area law

$$S \propto |\partial\mathcal{D}|$$

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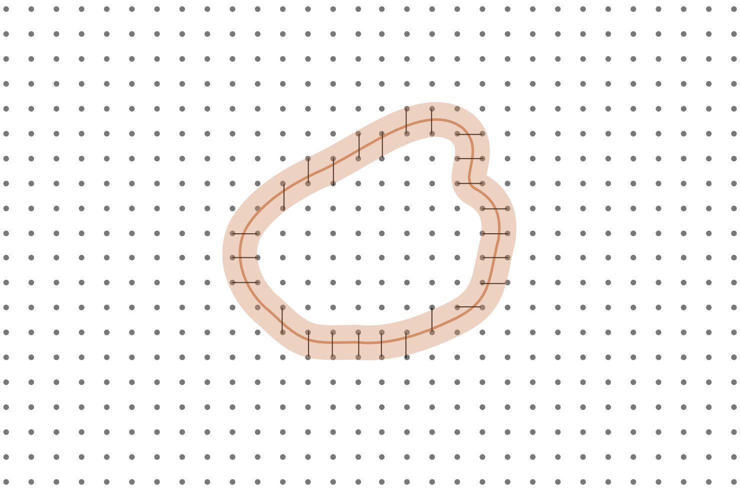

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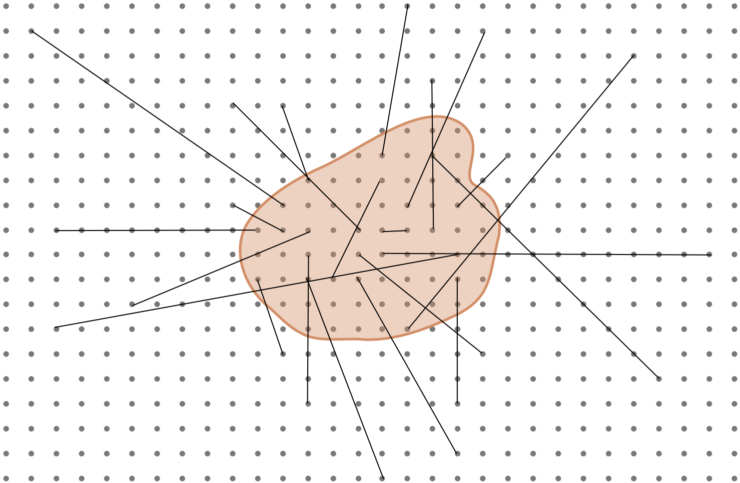
Typical states are strongly entangled



Random state

$|\psi\rangle = U_{\text{Haar}}|\text{trivial}\rangle$

Reduced density matrix

$\rho = \text{tr}_{\mathcal{D}^c}\left[|\psi\rangle\langle\psi|\right]$

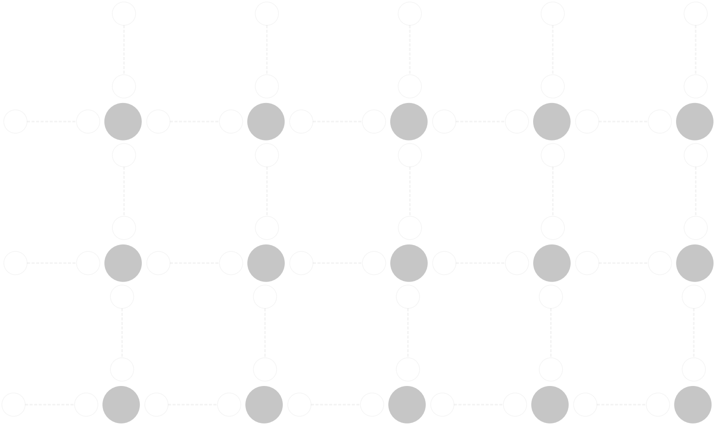
Entanglement entropy

$S = -\text{tr}\left[\rho \log \rho\right]$

Volume law

$$S \propto |\mathcal{D}|$$

Constructing weakly entangled states



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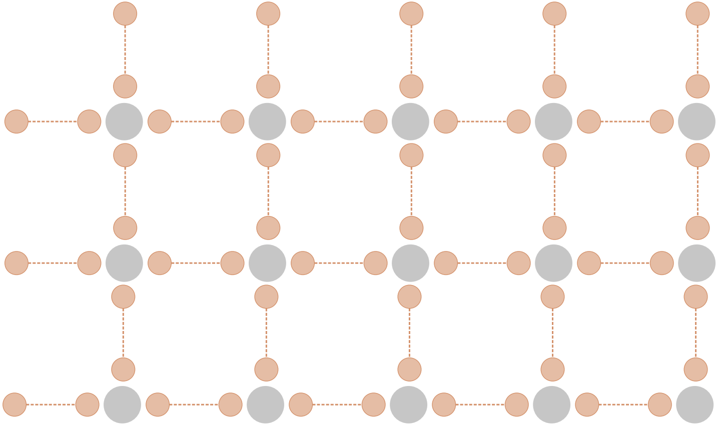
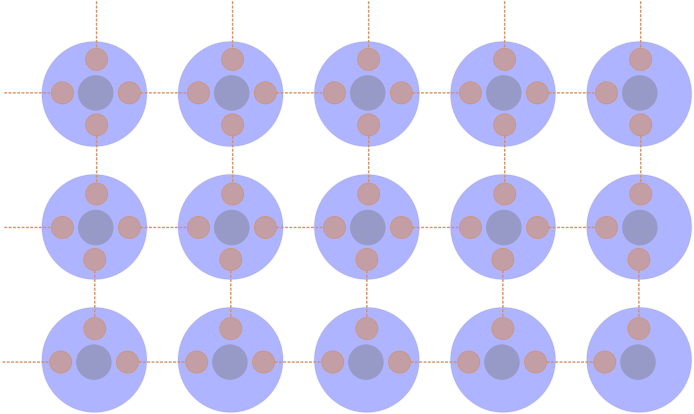


1. Put auxiliary **maximally entangled** states between sites

$$= \sum_{j=1}^{\chi} |j\rangle|j\rangle$$

Constructing weakly entangled states



1. Put auxiliary **maximally entangled** states between sites

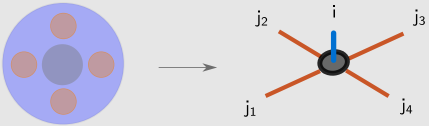
$$= \sum_{j=1}^{\chi} |j\rangle|j\rangle$$

2. Map to initial Hilbert space on each site

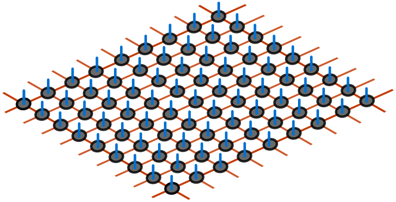
$$= A : \mathbb{C}^{4\chi} \to \mathbb{C}^{D}$$

Tensor network states: definition

Why "tensor" network?



$$A : \mathbb{C}^{4\chi} \to \mathbb{C}^{d} \qquad \longrightarrow \qquad A^{i}_{j_1,j_2,j_3,j_4}$$

$|A\rangle =$ 

with tensor contractions on links

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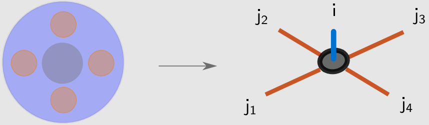

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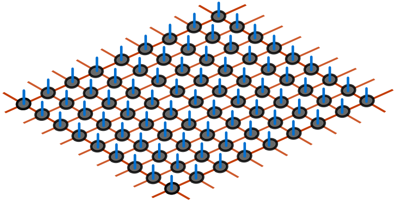
$|A\rangle =$ 

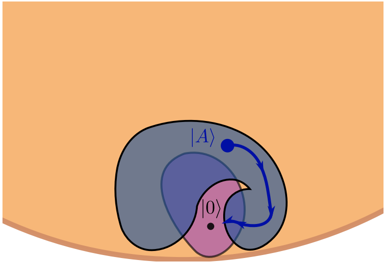
with tensor contractions on links

Optimization

Find best A for fixed χ $\quad (D \times \chi^4$ coeff.)

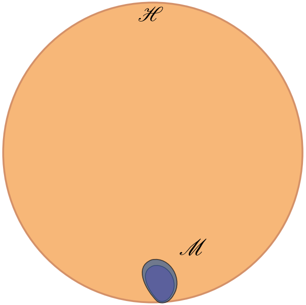
$$E_0 \simeq \min_A \frac{\langle A|\hat{H}|A\rangle}{\langle A|A\rangle}$$

for example go down $\frac{\partial E}{\partial A^i_{j_1, j_2, j_3, j_4}}$



Some facts

$d = 1$ spatial dimension



Theorems (colloquially)

1. For gapped H, tensor network states $|A\rangle$ approximate well $|0\rangle$ with χ fixed
2. **All** $|A\rangle$ are ground states of gapped H

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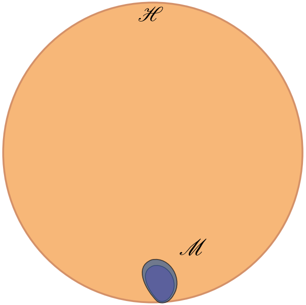
$d = 1$ spatial dimension



Theorems (colloquially)

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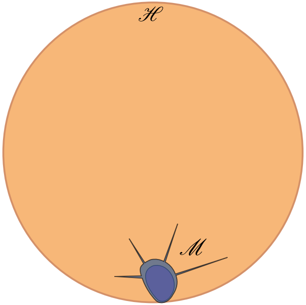
$d \geqslant 2$ spatial dimension



Folklore

1. For gapped H, tensor network states $|A\rangle$ approximate well $|0\rangle$ with χ fixed
2. **Most** $|A\rangle$ are ground states of gapped H

Limitations

Hard to contract in $d \geqslant 2$

In $d \geqslant 2$ one can have:

- $|A\rangle$ known
- $\langle A|\hat{\mathcal{O}}_i\hat{\mathcal{O}}_j|A\rangle$ hard to compute exactly

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Expressive but opaque

Generally hard to interpret

- Tensor carries IR-irrelevant information
- Hard to constrain long distance behavior

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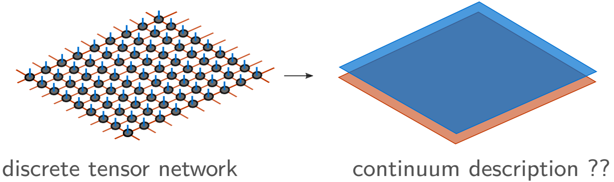
$\Longrightarrow$ Go to the continuum and **QFT**: Major objective and challenge



Matrix product states

Try to find the **ground state** of a spin chain, $H = \sum_j \hat{h}_j$:

$$|\psi\rangle = \sum_{i_1, i_2, \cdots, i_n} c_{i_1, i_2, \cdots, i_n} |i_1, \cdots, i_n\rangle$$

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Matrix Product States (MPS)

$$|A, L, R\rangle = \sum_{i_1, i_2, \cdots, i_n} \langle \mathrm{L}|A_{i_1}(1) A_{i_2}(2) \cdots A_{i_n}(n)|\mathrm{R}\rangle \, |i_1, \cdots, i_n\rangle$$

- A_i are $\chi \times \chi$ complex matrices
- A is a $2 \times \chi \times \chi$ tensor $[A_i]_{k,l}$
- $|L\rangle$ and $|R\rangle$ are χ-vectors.

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Remark: actually equivalent with the density matrix renormalization group (DMRG)

◊ $n \times 2 \times \chi^2$ parameters instead of 2^n

◊ χ is the **bond dimension** and encodes the size of the variational class

Recall the graphical notation

$$|A, L, R\rangle = \sum_{i_1, i_2, \cdots, i_n} \langle \mathrm{L}|A_{i_1}(1)A_{i_2}(2)\cdots A_{i_n}(n)|\mathrm{R}\rangle \; |i_1, \cdots, i_n\rangle$$

Notation: $[A_i]_{\mathbf{k,l}} =$  and k — $l = \sum \delta_{k,l}$ gives:

$|A, L, R\rangle =$ 

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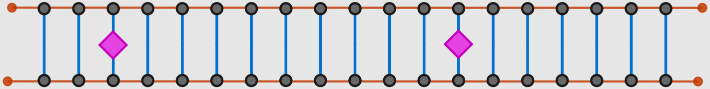
Example: computation of correlations

$\langle A|\mathcal{O}(i_k)\mathcal{O}(i_\ell)|A\rangle =$ 

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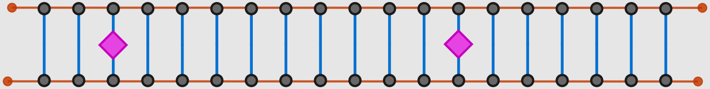
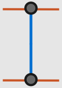
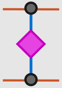
$\langle A|\mathcal{O}(i_k)\mathcal{O}(i_\ell)|A\rangle =$ 

can be done efficiently by iterating 2 maps:

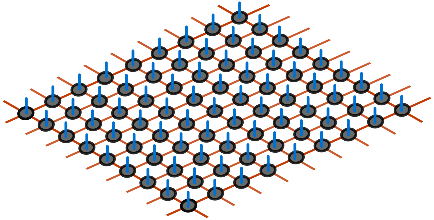
$\Phi =$  and $\Phi_{\mathcal{O}} =$ 

The contraction for a $d = 1$ system, can be seen as an open-system dynamics in $d = 0$.

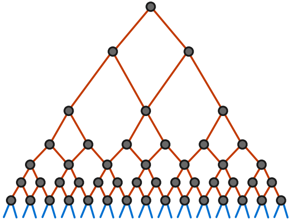
Generalizations: different tensor networks

Matrix Product States (MPS)

Projected Entangled Pair States (PEPS)

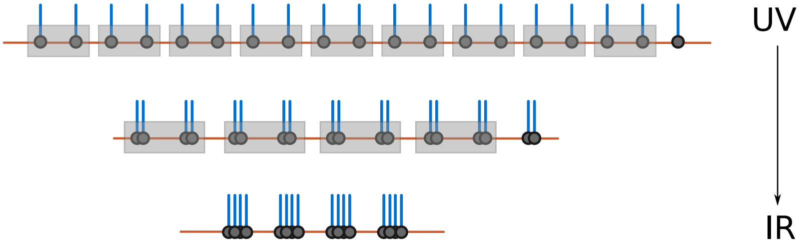


Multi-scale Entanglement Renormalization Ansatz (MERA)



Continuous Matrix Product States (cMPS)

Taking the continuum limit of a MPS



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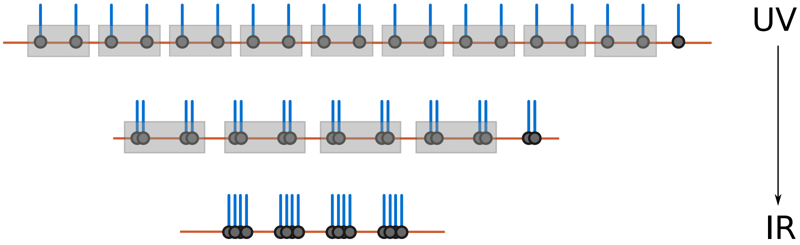

- the bond dimension χ stays fixed

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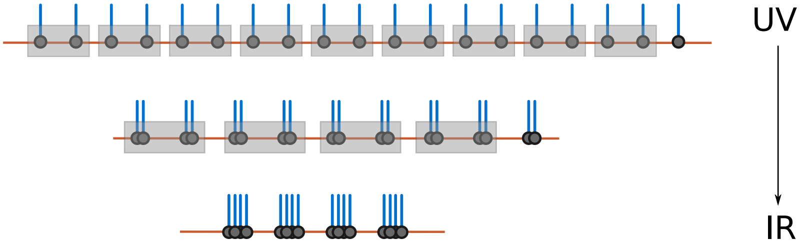


- the bond dimension χ stays fixed
- the local physical dimension explodes $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \longrightarrow \mathscr{F}(L^2([x, x+dx]))$.
  $\Longrightarrow$ **Spins** become **fields** – ($\simeq$ central limit theorem $\simeq$)

Continuous Matrix Product States

Type of ansatz for bosons on a fine grained $d = 1$ lattice

- Matrices $A_{i_k}(x)$ where the index i_k corresponds to $\psi^{\dagger i_k}(x)|0\rangle$ in physical space.

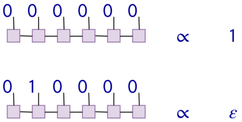
Informal cMPS definition

$$A_0 = \mathbb{1} + \varepsilon Q$$
$$A_1 = \varepsilon R$$
$$A_2 = \frac{(\varepsilon R)^2}{\sqrt{2}}$$
$$\dots$$
$$A_n = \frac{(\varepsilon R)^n}{\sqrt{n}}$$
$$\dots$$

so we go from ∞ to 2 matrices

Fixed by:

- Finite particle number



- Consistency

Continuous Matrix Product States

Definition

$$|Q, R, \omega\rangle = \langle \omega_L | \mathcal{P} \exp\left\{ \int_0^L dx \; Q \otimes \mathbb{1} + R \otimes \psi^\dagger(x) \right\} |\omega_R\rangle \; |0\rangle$$

- Q, R are $\chi \times \chi$ matrices,
- $|\omega_L\rangle$ and $|\omega_R\rangle$ are boundary vectors $\in \mathbb{C}^\chi$, for p.b.c. $\langle \omega_L | \cdot |\omega_R\rangle \to \mathrm{tr}[\,\cdot\,]$
- $[\psi(x), \psi^\dagger(y)] = \delta(x - y)$

Idea:

Continuous Matrix Product States

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Idea:

$$\begin{aligned} A(x) &\simeq A_0 \mathbb{1} + A_1 \psi^\dagger(x) \\ &\simeq \mathbb{1} \otimes \mathbb{1} + \varepsilon Q \otimes \mathbb{1} + \varepsilon R \otimes \psi^\dagger(x) \\ &\simeq \exp\left[\varepsilon \left(Q \otimes \mathbb{1} + R \otimes \psi^\dagger(x) \right)\right] \end{aligned}$$

Computations

Some correlation functions

$$\left\langle \hat{\psi}(x)^\dagger \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{TL} (R \otimes \overline{R}) \right]$$

$$\left\langle \hat{\psi}(x)^\dagger \hat{\psi}(0)^\dagger \hat{\psi}(0) \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{T(L-x)} (R \otimes \overline{R}) e^{Tx} (R \otimes \overline{R}) \right]$$

$$\left\langle \hat{\psi}(x)^\dagger \left[-\frac{d^2}{dx^2} \right] \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{TL} ([Q, R] \otimes [\overline{Q}, \overline{R}]) \right]$$

with $T = Q \otimes \mathbb{1} + \mathbb{1} \otimes \bar{Q} + R \otimes \bar{R}$

Example

Lieb-Liniger Hamiltonian

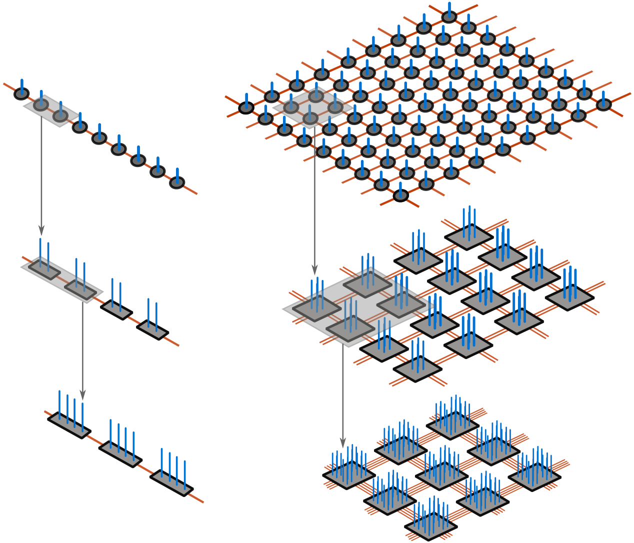
$$\mathcal{H} = \int_{-\infty}^{+\infty} dx \left[\frac{d\hat{\psi}^\dagger(x)}{dx} \frac{d\hat{\psi}(x)}{dx} + c\hat{\psi}^\dagger(x)\hat{\psi}^\dagger(x)\hat{\psi}(x)\hat{\psi}(x) \right]$$

Solve by **minimizing**:

$$\langle Q, R | \mathcal{H} | Q, R \rangle = f(Q, R)$$

with fixed particle density $\langle Q, R | \psi^\dagger(x) \psi(x) | Q, R \rangle$.

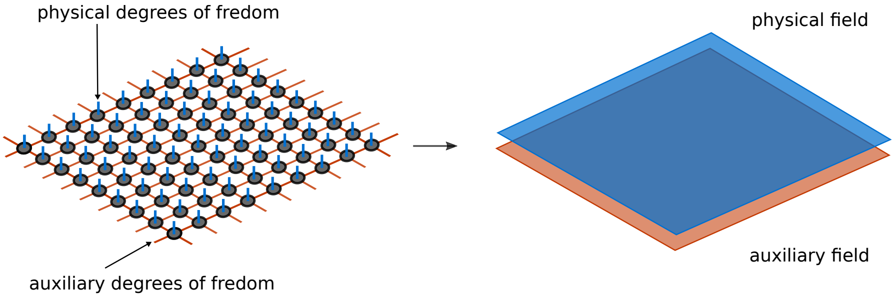
Continuous Tensor Networks: blocking



Upon **blocking**:

- The **physical** Hilbert space dimension D increases
- The **bond** (auxiliary space) dimension χ increases too

Result



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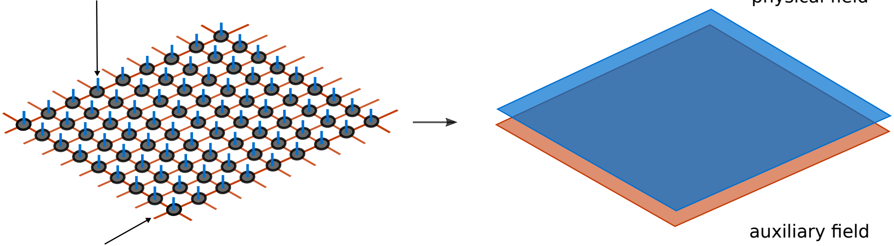

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Continuous tensor network state (heuristically)

State $|\alpha\rangle$ of $d+1$ QFT from an auxiliary d dimensional theory of random fields ϕ:

$$|\alpha\rangle = \int \mathcal{D}\phi \; \exp\left\{ - \int \mathrm{d}^d x \; \mathcal{L}[\phi(x)] - \alpha[\phi(x)] \underbrace{\hat{\psi}^\dagger(x)}_{\text{creation}} \right\} |\Omega\rangle$$

1. Genuine continuum limit of discrete tensor networks
2. The toolbox is translated to the continuum

Choice of tensor around which to expand...

For **MPS**, not much choice:

 $=$ $+ \ \varepsilon \cdots$

$$= \mathbb{1} \otimes |0\rangle + \varepsilon\, Q \otimes |0\rangle + \varepsilon\, R \otimes \psi^\dagger(x)|0\rangle$$

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1. Take a δ between all legs $\sim$ GHZ state $T^{(0)} =$ 
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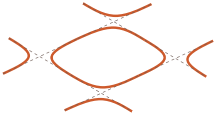
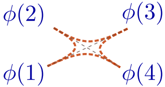
3. Take the sum of pairs of identities in both directions $T^{(0)} =$  $+$ 



Ansatz

1 – Take a "Trivial" tensor:

$$T^{(0)}_{\phi(1),\phi(2),\phi(3),\phi(4)} =$$



$$\sim \exp\left\{ \frac{-1}{2} \sum_{k=1}^{D} [\phi_k(1) - \phi_k(2)]^2 + [\phi_k(2) - \phi_k(3)]^2 + [\phi_k(3) - \phi_k(4)]^2 + [\phi_k(4) - \phi_k(1)]^2 \right\}$$

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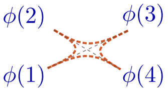

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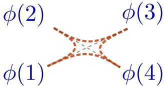

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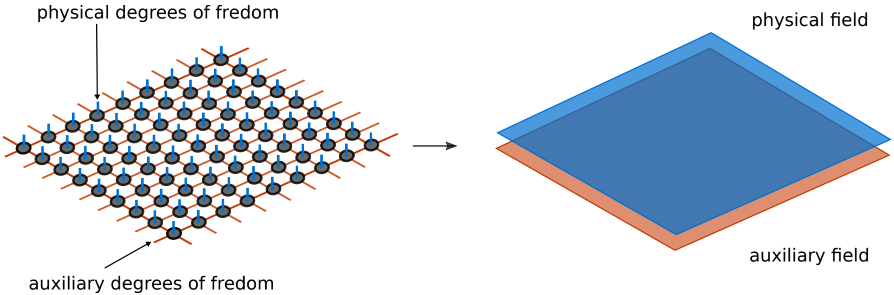
3 – Realize tensor contraction = functional integral and trivial tensor gives free field measure.

Result



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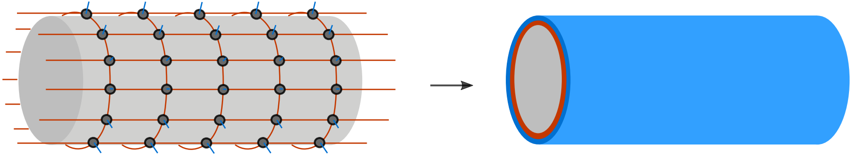
Operator definition



$$|V,\alpha\rangle =$$

$$\mathrm{tr}\left[\mathcal{T}\exp\left(-\int_0^T d\tau \int_S dx\, \frac{\hat{\pi}_k(x)\hat{\pi}_k(x)}{2} + \frac{\nabla\hat{\phi}_k(x)\nabla\hat{\phi}_k(x)}{2} + V[\hat{\phi}(x)] - \alpha[\hat{\phi}(x)]\,\psi^\dagger(\tau,x)\right)\right]|0\rangle$$

where:

- $\hat{\phi}_k(x)$ and $\hat{\pi}_k(x)$ are χ independent canonically conjugated pairs of (auxiliary) field *operators*: $[\hat{\phi}_k(x),\hat{\phi}_l(y)]=0$, $[\hat{\pi}(x)_k,\hat{\pi}_l(y)]=0$, and $[\hat{\phi}_k(x),\hat{\pi}_l(y)]=i\delta_{k,l}\,\delta(x-y)$ acting on a space of $d-1$ dimensions.

Wave-function definition

A generic state $|\Psi\rangle$ in Fock space can be written:

$$|\Psi\rangle = \sum_{n=0}^{+\infty} \int_{\Omega^n} \frac{\varphi_n(x_1, \cdots, x_n)}{n!} \psi^\dagger(x_1) \cdots \psi^\dagger(x_n) |0\rangle$$

where ϕ_n is a symmetric n-particle wave-function

Functional integral representation

$$\varphi_n(x_1, \cdots, x_n) = \langle\, \alpha[\phi(x_1)] \cdots \alpha[\phi(x_n)] \,\rangle_{\text{aux}}$$

with:

$$\langle \cdot \rangle_{\text{aux}} = \int \mathcal{D}\phi \;\cdot\; B(\phi|_{\partial\Omega}) \exp\left[-\frac{1}{2} \int_{\Omega} \mathrm{d}^d x \, [\nabla \phi_k(x)]^2 + V[\phi(x)]\right]$$

- $\sim$ Moore-Read wave-function for Quantum Hall, but generic QFT

Expressivity and stability

How big are cTNS?

Stability

The sum of two cTNS of bond field dimension χ_1 and χ_2 is a cTNS with bond field dimension $\chi \leqslant \chi_1 + \chi_2 + 1$:

$$|V_1, \alpha_1\rangle + |V_2, \alpha_2\rangle = |W, \beta\rangle$$

Expressiveness

All states in the Fock space can be approximated by cTNS:

- A field coherent state is a cTNS with $\chi = 0$
- Stability allows to get all sums of field coherent states

Note: expressiveness can also be obtained with $\chi = 1$ but it is less natural. Flexibility in χ makes the expressivity higher for restricted classes of V and α.

Computations

Define generating functional for normal ordered correlation functions

$$\mathcal{Z}_{j',j} = \frac{1}{\langle V,\alpha|V,\alpha\rangle}\langle V,\alpha|\exp\left(\int \mathrm{d}x\, j'(x)\psi^\dagger(x)\right)\exp\left(\int \mathrm{d}x\, j(x)\psi(x)\right)|V,\alpha\rangle$$

Operator representation

$$\mathcal{Z}_{j'j} = \mathrm{tr}\left[B\otimes B^* \mathcal{T}\exp\left\{\int_{-T/2}^{T/2}\left(\mathrm{T}_{j'j} - \int_S j\cdot j'\right)\right\}\right]$$

with **transfer matrix**:

$$\mathrm{T}_{j'j} = \int_S \mathrm{d}x\; \mathcal{H}(x)\otimes \mathbb{1} + \mathbb{1}\otimes\mathcal{H}^*(x) + \left(\alpha[\hat{\phi}(x)] + j'(x)\right)\otimes\left(\alpha[\hat{\phi}(x)]^* + j(x)\right)$$

and

$$\mathcal{H}(x) = \sum_{k=1}^{D}\frac{[\hat{\pi}_k(\mathbf{x})]^2 + \left[\nabla\hat{\phi}_k(\mathbf{x})\right]^2}{2} + V[\hat{\phi}(x)]$$

$\Longrightarrow$ cMPS brought us from 1 to 0, cTNS bring us from d to $d-1$.

Renormalization

Scaling

- $d = 2$, All powers of the field in V and α yield relevant couplings
- $d = 3$, The powers $p = 1, 2, 3, 4, 5$ of the field in V yield relevant $\Delta > 0$ couplings. $p = 6$ is marginal in V. For α, $p = 1, 2$ are relevant and $p = 3$ is marginal. All other p are irrelevant.

For finite bond field dimension in $d = 3$, finite number of parameters for **renormalized** cTNS:

$$V(\phi) = A\phi + B\phi\phi + C\phi\phi\phi + D\phi\phi\phi\phi + E\phi\phi\phi\phi\phi + F\phi\phi\phi\phi\phi\phi$$
$$\alpha(\phi) = X\phi + Y\phi\phi + Z\phi\phi\phi$$

Proper renormalization procedure not checked yet

Generalization

For a general Riemanian manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, define:

$$|V, B, \alpha\rangle = \int \mathcal{D}\phi \, B(\phi|_{\partial\mathcal{M}}) \exp\left\{-\int_{\mathcal{M}} \mathrm{d}^d x \sqrt{g}\left(\frac{g^{\mu\nu}\partial_\mu\phi_k\partial_\nu\phi_k}{2} + V[\phi, \nabla\phi] - \alpha[\phi, \nabla\phi]\,\psi^\dagger\right)\right\}|0\rangle$$

i.e. add curvature and possible anisotropies in V and α

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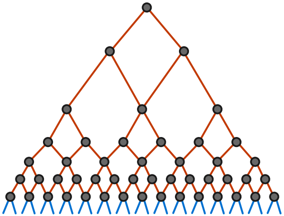
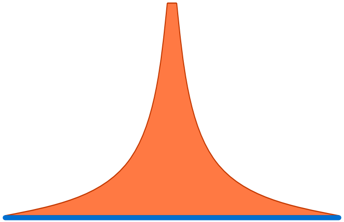
Example: $\alpha[x, \phi, \nabla\phi]$ localized on the boundary and hyberbolic metrix g:





$\rightarrow$ **cMERA**-like in $d-1$ dimensions

Future

Limitations and work for the future

- Quite formal out of the Gaussian regime (but Gaussian still non-trivial)
- Computation through dimensional reduction quite hard to carry
- Limited to bosonic field theories (so far)
- Gauge invariant states
- Can one say anything about topology?

Summary

$$|V, B, \alpha\rangle = \int \mathcal{D}\phi \, B(\phi|_{\partial\Omega}) \exp\left\{-\int_{\Omega} \mathrm{d}^d x \, \frac{1}{2} \sum_{k=1}^{D} [\nabla\phi_k(x)]^2 + V[\phi(x)] - \alpha[\phi(x)] \, \psi^\dagger(x)\right\} |0\rangle$$

Continuous tensor network states are natural continuum limits of tensor network states and natural higher d extensions of continuous matrix product states.

1. Obtained from discrete tensor networks
2. Can be made Euclidean invariant
3. **Motto of tensor networks**: trade a dimension for a variational optimization
4. Still need to be properly renormalized (in perturbative and RG sense)
5. Still needs to be used to approximate non-trivial non-Gaussian ground states