What's the deal with Quantum Field Theory?

like really, what is it?

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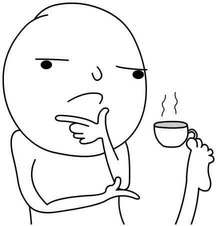
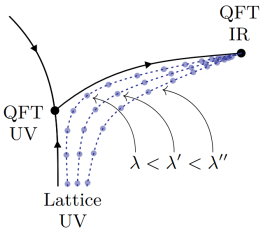




MPQ Theory workshop on lock-down

May 7th, 2020

MPQ

Why talk about QFT?

It is timely:

- Recently growing interest in the group
- Until not long ago, I understood essentially nothing

Interesting developments in an old subject:

- ϕ^4_2 solved to good precision "Rychkov challenge" (got me interested)
- ϕ^4_4 officially dead.

Why QFT is usually poorly explained

- No separation between the definition of the object and the computation tool
- Only perturbation theory, with everything blowing up
- Dirac fermions and massless vector bosons introduce orthogonal complications $\rightarrow \gamma^\mu \gamma^\nu \gamma_0 \gamma_5 + \cdots$
- It is unclear what is not known at all, and what we do not do just because we want to spare the ϵ's and δ's
- All the QFTs presented apart (hopefully) from QCD do not exist

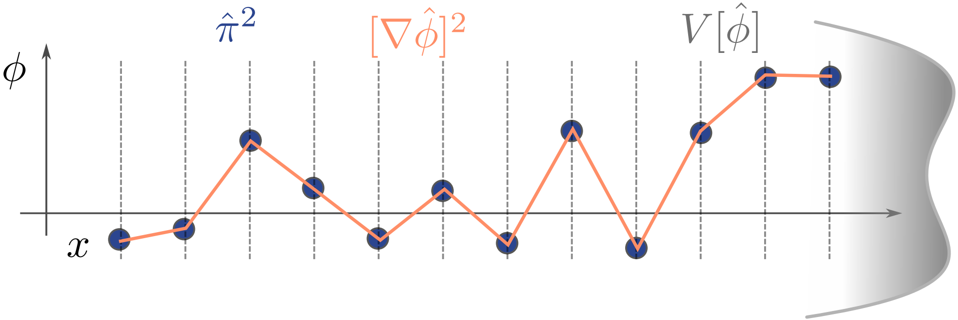
Ergo, the subject of the talk

What's the deal with QFT? What is known mathematically? What is not known? What is hard to compute? What is hard to define? What has been done? What is yet to be done?

Outline

1. Intuitive definitions
2. The free difficulties
3. The interacting difficulties
4. Axiomatic and constructive field theory
5. Example of ϕ^4
6. How to compute stuff

Intuitive definition: canonical quantization



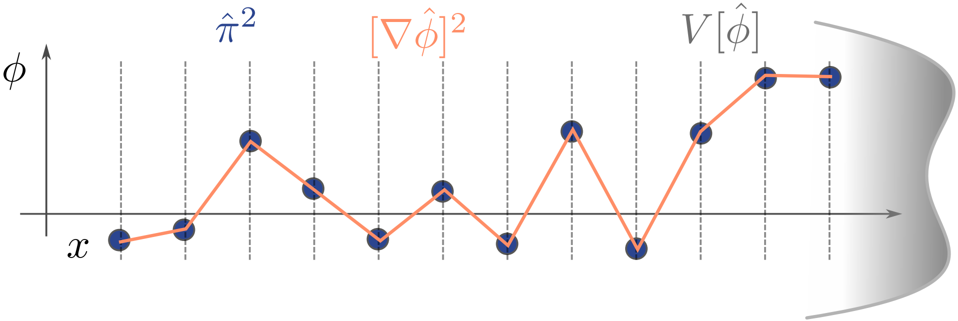
Hamiltonian

A continuum of nearest neighbor coupled anharmonic oscillators

$$\hat{H} = \int_{\mathbb{R}^d} \mathrm{d}^d x \quad \underbrace{\frac{\hat{\pi}(x)^2}{2}}_{\text{on-site inertia}} + \underbrace{\frac{[\nabla\hat{\phi}(x)]^2}{2}}_{\text{spatial stiffness}} + \underbrace{V(\hat{\phi}(x))}_{\text{on-site potential}}$$

with canonical commutation relations $[\hat{\phi}(x), \hat{\pi}(y)] = i\delta^d(x-y)\mathbb{1}$ (*i.e.* bosons)

Intuitive definition



Hilbert space

Fock space $\mathscr{H}_{\text{QFT}} = \mathscr{F}[L^2(\mathbb{R}^d)]$ – just like $x, p \to (a, a^\dagger)$ do $\hat{\pi}, \hat{\phi} \to \hat{\psi}, \hat{\psi}^\dagger$

$$|\Psi\rangle = \sum_{n=0}^{+\infty} \int \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \, \underbrace{\varphi_n(x_1, x_2, \cdots, x_n)}_{\text{wave function}} \underbrace{\hat{\psi}^\dagger(x_1)\hat{\psi}^\dagger(x_2)\cdots\hat{\psi}^\dagger(x_n)}_{\text{local oscillator creation}} |\text{vac}\rangle$$

Intuitive definition: functional integral

Insert $\mathbb{1} = \int \mathcal{D}\phi \, |\phi\rangle\langle\phi|$ in expression for correlation functions and $t = i\tau$ gives

Functional integral representation

Representation of correlation functions in terms of random fields

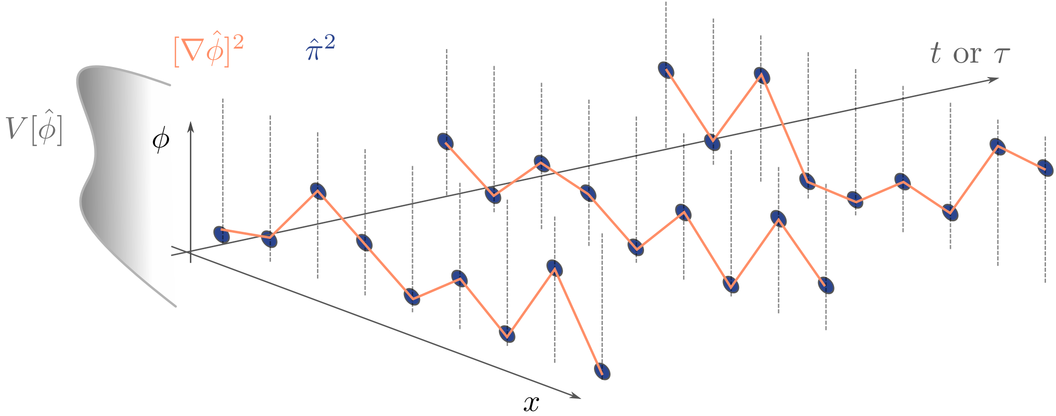
$$\langle 0|\hat{\phi}(\tau_1, x_1) \cdots \hat{\phi}(\tau_n, x_n)|0\rangle := \int \phi(\tau_1, x_1) \cdots \phi(\tau_1, x_n) \; e^{-S(\phi)} \underbrace{\mathcal{D}\phi}_{\text{"Lebesgue measure"}}$$

with the action / weight where $\hat{\pi} \to \frac{d\phi}{d\tau}$

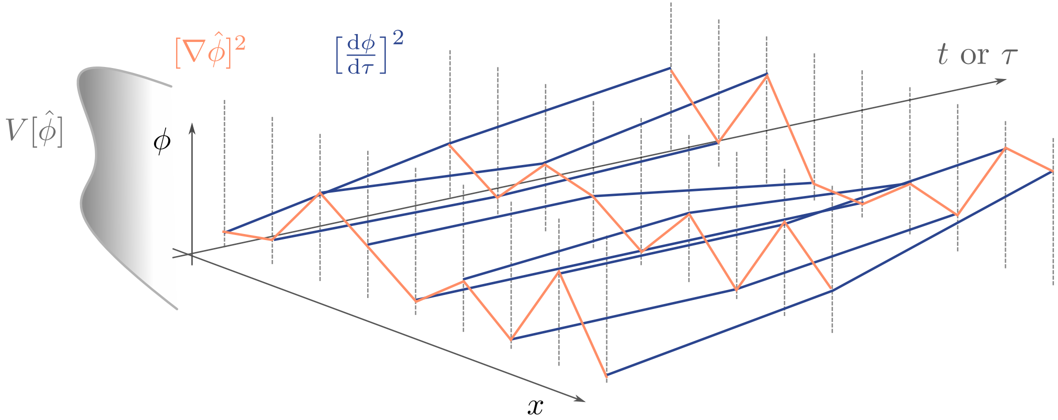
$$S(\phi) = \int d^d x \, d\tau \quad \underbrace{\frac{1}{2}\left[\frac{d\phi}{d\tau}\right]^2}_{\text{inertia a.k.a time stiffness}} + \underbrace{\frac{[\nabla\phi]^2}{2}}_{\text{spatial stiffness}} + \underbrace{V(\phi)}_{\text{on-site potential}}$$

Inertia = time stiffness $\Longrightarrow$ Euclidean rotation invariance $\Longrightarrow$ Lorentz

Intuitive definition: functional integral



Intuitive definition: functional integral



What are the problems - Hilbert space approach

The Hamiltonian is ill defined on all states in the Hilbert space because of infinite zero point energy *i.e.* terms $\propto \hat{\psi}(x)\hat{\psi}^\dagger(x)$

$$\langle \Psi_1 | \hat{H} | \Psi_2 \rangle = \pm\infty \quad \text{and even} \quad \langle \text{vac} | \hat{H} | \text{vac} \rangle \propto \delta^d(0) = +\infty$$

If the divergent vacuum terms are removed, the Hamiltonian is not bounded from below

$$\forall \; |\Psi\rangle \in \mathscr{H}, \; \langle \Psi | \hat{H}_{\text{finite}} | \Psi \rangle = \text{finite but} \;\; \exists \; \Psi_n \text{ s.t.} \lim_{n\to+\infty} \langle \Psi_n | H_{\text{finite}} | \Psi_n \rangle = -\infty$$

and worse

$$|0\rangle := \lim_{n\to+\infty} |\Psi_n\rangle \notin \mathscr{H}$$

What are the problems - Functional integral approach

Many issues, related to the fact that there is no Lebesgue measure $\mathcal{D}\phi$ on functions [definition issue], and *no equivalent* for $d \geqslant 2$ [real world issue]

The field is not even a function

Entropy dominates energy

$$\langle \phi(x)^2 \rangle = \int \mathrm{d}^d p \, \frac{1}{m^2 + p^2} = +\infty \quad \text{if } d \geqslant 2$$

We penalize irregular and large ϕ, yet the only ones that "typically" occur are so irregular and large the penalty term is ill defined.

How are they are solved in the free case - Hamiltonian

Bogoliubov transform

Go from $\hat{\psi}(x), \hat{\psi}^\dagger(x)$ to $a(p), a^\dagger(p)$ with

$$a(p) = \frac{1}{\sqrt{2}}\left(\sqrt{\omega_p}\,\hat{\phi}(p) + \frac{\hat{\pi}(p)}{\sqrt{\omega_p}}\right) \quad \text{with} \quad \omega_p = \sqrt{p^2 + m^2}$$

which yields

$$H = \int \mathrm{d}p\,\omega_p\,\frac{1}{2}\left(a_p^\dagger a_p + a_p a_p^\dagger\right)$$

Solution

- Take $H_{\mathrm{QFT}} \equiv \,:H:_a$
- $|\text{ground state}\rangle = |\text{vacuum}\rangle_a$
- $\mathscr{H}$ built from $a_{p_1}^\dagger \cdots a_{p_n}^\dagger |\text{vacuum}\rangle_a$

It is easy to define what you can exactly solve: **take the solution as starting point**

Quick note $\hat{\psi}(x)$ vs $a(x)$

Careful, they are different

[listen to me]

How are they solved in the free case - functional integral

It is difficult to define a measure on functions. The trick is to define not Lebesgue but Gaussian:

Wiener measure ($d = 1$)

The measure

$$d\mu(\phi) \text{ “} := \text{” } \exp\left[-\frac{1}{2}\int(\partial_x\phi)^2\right]\mathcal{D}\phi$$

can be defined rigorously in $d = 1$ and is supported on $C^{1/2}$ functions. In fact, ϕ is the Brownian motion.

Even works for interacting:

$$d\mu_\lambda(\phi) := \exp\left[-\lambda\int\phi^4\right]d\mu(\phi)$$

is perfectly well defined (peculiarity of $d = 1$).

How are they solved in the free case - functional integral

For $d \geqslant 2$, no measure on functions since ϕ is not even a function. As before start from solution.

Bochner-Milnos theorem

Take a distribution $D(x, y)$ that has reasonable properties of a correlation function (positive, symmetric, not too weird), then there exists a Gaussian process ϕ of which it is the correlation function:

$$D(x, y) = \langle \phi(x)\phi(y) \rangle$$

ϕ is a distribution valued random variable

So, to do things properly:

1. Solve the functional integral dirty (removing infinities, using black magic)
2. Use the found 2-point correlation as starting point to define the theory

How about interactions?

Use the free theory that is understood + perturbation theory

$$\int d\mu(\phi) \exp\left[-\int \lambda\phi^4\right] \simeq \int d\mu(\phi) \left[1 + \lambda \int \phi^4(x) + \frac{1}{2}\lambda^2 \iint \phi^4(x)\phi^4(y) + \cdots\right]$$

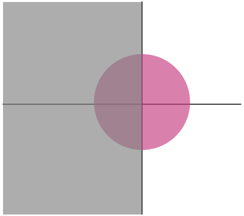
2 difficulties

- Each term in the series in infinite [need regularization]
- Removing/smoothing the infinities term by term, the series is divergent

Divergence of the expansion

First noted by Dyson in a 2 page PRL

any physical quantity $= f(g) \underset{\text{illicit}}{=} \sum_n a_n g^n$ diverges $\forall\ g$



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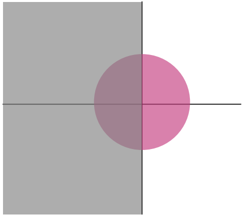


$$\text{any physical quantity} = f(g) \underset{\text{illicit}}{=} \sum_n a_n g^n \quad \text{diverges } \forall\, g$$

For ϕ_0^4

$$\int_{\mathbb{R}} \mathrm{d}\phi \;\exp(-m^2\phi^2 - g\phi^4) \underset{\text{illicit}}{=} \sum_{n=0}^{+\infty} \frac{(-g)^n}{n!} \int_{\mathbb{R}} \mathrm{d}\phi\, \phi^{4n} \exp(-m^2\phi^2)$$

$$= \sum_{n=0}^{+\infty} \frac{(-g)^n}{n!\, m^{2n+1/2}} \int_0^{+\infty} \mathrm{d}u\, u^{2n+1/2} \exp(-u)$$

$$= \sum_{n=0}^{+\infty} \frac{(-g)^n}{m^{2n+1/2}} \frac{\Gamma(2n+3/2)}{\Gamma(n+1)}$$

So do QFT even exist? Are they needed?

Effective field theories as the only thing

All theories have a **UV** (short distance) cutoff. Some approximate field theory description makes sense far (but not infinitely far) from the cutoff.

but two things:

- A regulated QFT is not longer a reasonable QFT (either non-local, non-relativistic, unstable vacuum, etc.). The theory underlying the effective QFT would have to be different (Strings?)
- Seeing all QFT as effective QFT is **not needed**. **Could if be QFT all the way down?**

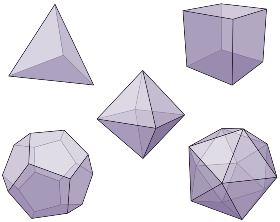
Ways forward

Start from something simple so it becomes simple

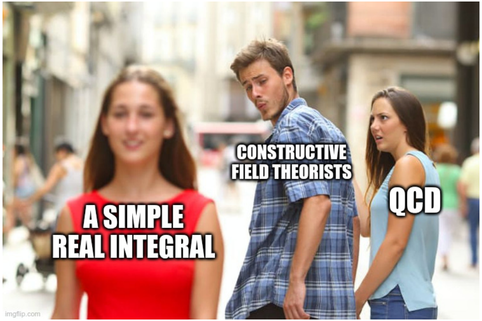
The idea of constructive field theory. Try to make sense of the ϕ^4 term in $d = 2$, then climb your way up to real stuff (QCD).

Make it complicated so it becomes simple

What high energy theorists do. Make it highly complicated such that the problems cancel out, things can be solved almost exactly, and proceed as with the free field (take the final point as definition).



Ways forward, but in memes



Axiomatic (Lorentzian) QFT

Wightman functions

Imagine you have a QFT, *e.g.* in canonical quantization. Then the Wightman functions are

$$f_n(x_1, x_2, \cdots, x_n) := \langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle$$

and are tempered distributions.

Wightman reconstruction theorem

Imagine you are given **Wigthman functions** $f_1, f_2, \cdots, f_n, \cdots$ that are reasonable, in the sense that they verify the sort of things correlation functions from QFT should (locality, Lorentz invariance, microcausality), a.k.a the **Wightman axioms** then there exists a QFT of which they are the correlation functions

Arthur Strong Wightman

(1922 – 2013)

Axiomatic Euclidean QFT

Schwinger functions

Imagine you have a QFT, *e.g.* in canonical quantization. Then the Schwinger functions are the correlation functions in imaginary time:

$$S_n(x_1, \cdots, x_n) := \langle \phi(x_1, \tau_1) \cdots \phi(x_n, \tau_n) \rangle$$

and are not a priori as well behaved.

Osterwalder-Schrader reconstruction

Imagine you are given **Schwinger functions** $S_1, S_2, \cdots, S_n, \cdots$ that are reasonable + some non-obvious technical conditions, a.k.a the **Osterwalder-Schrader axioms** then they can be analytically continued to Wightman functions

Robert Schrader
(1939 – 2015)

Konrad Osterwalder
(born 1942)

Constructive field theory

Idea in a nutshell

Start from a random field on a lattice of size a with probability distribution:

$$dP(\phi) = \exp(-\mathscr{S}^a(m_a, \lambda_a, \phi))\, d\phi$$

and try to control the continuum limit of the Schwinger functions by tuning m_a, λ_a, then declare that :

$$S_n := \lim_{a \to 0} S_n^a$$

and then try to prove that the limit verifies the O-S axioms.

Glimm, Jaffe, Fröhlich, Sokal, Kupiainen, Gawedski, Rivasseau, Sénéor, Chatterjee $\cdots$

Lattice ϕ^4

Define the probability measure $d\mu(\phi) = \exp(-S(\phi)) \prod_{i \in \text{lattice}} d\phi_i$

$$S(\phi) = \sum_{\langle i,j \rangle} \underbrace{\frac{(\phi_i - \phi_j)^2}{2a^2}}_{(\nabla\phi)^2/2} a^d + \sum_i \frac{1}{2}\mu_a^2 \phi_i^2 + \frac{1}{4}\lambda_a \phi_i^4$$

Continuum limit

Send a to 0, while tuning λ_a and μ_a such that

$$\langle \phi(x_{i_1})\phi(x_{i_2})\cdots\phi(x_{i_n}\rangle \to f(x_1, x_2, \cdots, x_n) \quad \text{non-trivial}$$

intuitively we should take $\lambda_a = a^{-[\lambda]}\lambda$ and $\mu_a = a^{-[\mu]}\mu$ but not the right scaling

ϕ^4_0: trivially exists and exactly solvable

Recall that

$$\int_{\mathbb{R}} d\phi \ \exp(-m^2\phi^2 - g\phi^4) \underset{\text{illicit}}{=} \sum_{n=0}^{+\infty} \frac{(-g)^n}{m^{2n+1/2}} \frac{\Gamma(2n+3/2)}{\Gamma(n+1)}$$

But of course the integral exists, and can be computed (Simpson) or *e.g.*

$$\begin{aligned}\int_{\mathbb{R}} d\phi \ \exp(-m^2\phi^2 - g\phi^4) &\underset{\text{licit}}{=} \sum_{n=0}^{+\infty} \frac{(-m^2)^n}{n!} \int_{\mathbb{R}} d\phi \, \phi^{2n} \exp(-g\phi^4) \\ &= \sum_{n=0}^{+\infty} \frac{(-m^2)^n}{g^{n+1/2}} \frac{1}{2} \int_0^{+\infty} dv \, v^{n/2+1/4} \exp(-u) \\ &= \frac{1}{2} \sum_{n=0}^{+\infty} \frac{(-m^2)^n}{g^{n+1/2}} \frac{\Gamma(n/2+5/4)}{\Gamma(n+1)} \quad \underset{R_c=+\infty}{\text{absolutely conv.}}\end{aligned}$$

ϕ^4_1: clearly exists, solvable to arbitrary precision

A field in 0 space and 1 time dimension is just an an-harmonic oscillator.

Solve the corresponding 1-body Schrödinger equation, *e.g.* with exact diagonalization on a truncated mode basis.

ϕ^4_2: exists, not easy to solve

First non-trivial example to have been rigorously constructed

$$S(\phi) = \sum_{\langle i,j \rangle} \underbrace{\frac{(\phi_i - \phi_j)^2}{2a^2}}_{(\nabla\phi)^2/2} a^2 + \sum_i \frac{1}{2}\mu_a^2\phi_i^2 + \frac{1}{4}\lambda_a\phi_i^4$$

Taking the limit

The right way to get the continuum limit is to take:

$$\mu_a = \mu a^2 + \frac{3}{2}\log(a)a^2\lambda$$
$$\lambda_a = \lambda a^2$$

which is equivalent to normal ordering the interaction term.

Basically, at first order in perturbation theory, the ϕ^4 term behaves like a ϕ^2 term times a log divergent constant.

ϕ^4_3: exists, not easy to solve

Same reasoning, but more complicated scaling since 2 divergent diagrams in perturbation theory.

Taking the limit

The right way to get the continuum limit is to take:

$$\mu_a = a^{-[\mu]}(\mu + C_1 \lambda a^{-1} + C_2 \lambda^2 \log(a))$$
$$\lambda_a = \lambda a^{-[\lambda]}$$

which is equivalent to normal ordering the interaction term.

In general

In general the continuum limit requires typically that the mass term is a series of the coupling:

$$\mu_a = a^{-[\mu]}\mu + a^{-[\mu]} \sum_n \lambda^n f_n(a)$$

where the $f_n(a)$ diverge when $a \to 0$.

Several options:

- **Non-renormalizable** Infinitely many $f_n(a)$ are non zero, and can be fixed arbitrarily
- **Just renormalizable** Infinitely many $f_n(a)$ are non zero, but are determined once a finite number of parameters are fixed
- **Super renormalizable** Only a finite number of $f_n(a)$ are non zero (can be found from perturbation theory)

When just renormalizable, it could be that the series of $f_n(a)$ diverges...

Understanding the need for renormalization

Main reason

For interacting quantum field theories, the naive way to take the continuum limit, using engineering dimensions, is wrong, because the field takes larger and larger values as we get close to the continuum limit.

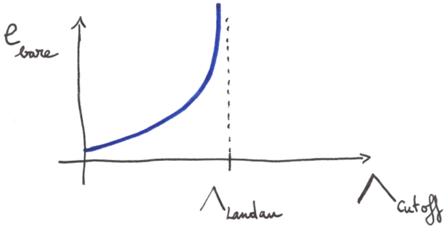
Some sad facts

ϕ^4_4 does not exist (is trivial)

Almost proved for a long time
Proved by Aizenman & Duminil-Copin, 2020

***QED* does not exist (Landau pole)**

Very suspected to be true (numerics and RG)



Some open problems

Millennium prize

For some compact Lie group G, construct a QFT verifying a set of axioms as strong as Wightman's that gives the "standard" Yang-Mills (non-abelian Gauge theory) perturbation expansion when Taylor expanded.

Still worth a Fields medal

Rigorously construct **one** example of non-trivial scalar QFT in $3+1$ dimensions, or prove that it is impossible

For another time, how to compute stuff?

Once one understands what a QFT actually is, most condensed matter techniques that we know can be used.

For example, for ϕ_2^4, critical coupling $f_c = \lambda/\mu^2$

Method	$f_c^{\text{cont.}}$	Year	Ref.
Tensor network coarse-graining	10.913(56)	2019	[9]
Borel resummation	11.23(14)	2018	[6]
Renormalized Hamil. Trunc.	11.04(12)	2017	[5]
Matrix Product States	11.064(20)	2013	[7]
Monte Carlo	11.055(20)	2019	[15]
This work	11.0861(90)	2020	

TABLE I. Comparison of several estimates of the critical coupling constant $f_c^{\text{cont.}}$ in the continuum obtained using different methods.