Variational optimisation in relativistic QFT

without *any* cutoff

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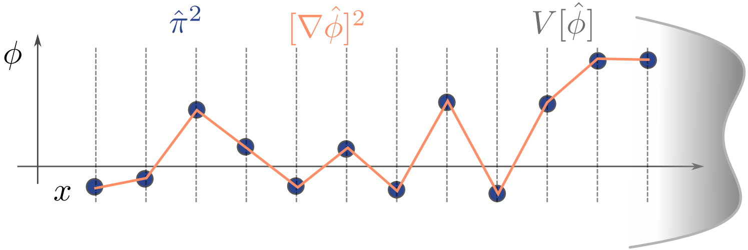


MPQ Many-Body-QFT group meeting
December 3rd, 2020

MPQ

Fundamental Physics with tensor networks

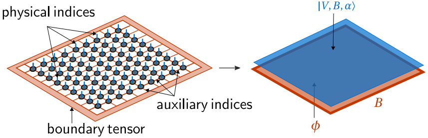
To apply tensor networks to fundamental theories, we need to understand:

1. Weird degrees of freedom (Gauge theories)
2. The continuum
3. (Relativistic Hamiltonians)

What we did so far on the continuum

“Analytical” Continuous tensor networks

1. Introduce a “good” definition of continuous tensor network (with Ignacio)



2. Show that in a simple setup it does the job (with Teresa and Patrick)

$\longrightarrow$ both non-relativistic, “condensed-matter QFT”

“Numerical” Continuous tensor networks

1. Discretize ϕ^4_2 on a super-fine lattice, solve with standard methods, extrapolate the result to the continuum limit (with Clément)

True vs Effective QFT

Against the "why bother since there is always a cutoff?"

Effective QFT

The theory has a momentum/energy cutoff Λ large but finite $\Lambda \gg m$, where m is the gap.

The fundamental theory is not known, but in perturbation theory, one can take $\Lambda \to \infty$ term by term to get a good approximation of physics at scale m.

Examples

1. QED with matter
2. ϕ_4^4

True QFT

The limit $\Lambda \to +\infty$ can be taken exactly, and the theory is valid "all the way down".

All quantities exist non-perturbatively in the limiting theory, for arbitrarily high energy. No cutoff whatsoever in principle.

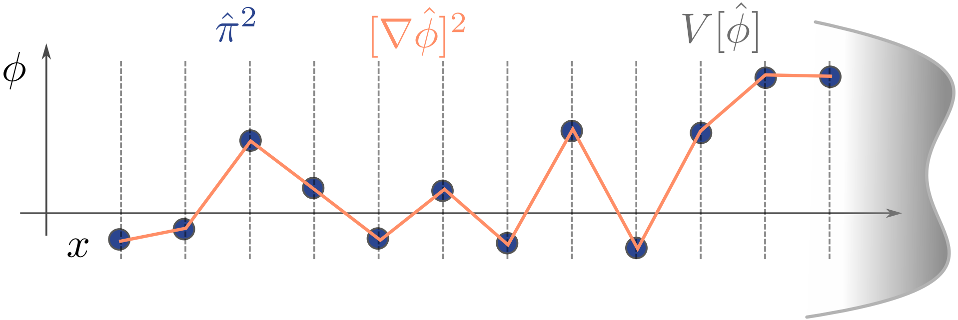
Examples

1. QCD without too much matter
2. ϕ_2^4 and ϕ_3^4
3. Sine-Gordon, Gross-Neveu, etc.

Outline

1. ϕ^4 theory – the condensed matter way
2. Divergences and standard resolution
3. ϕ^4 theory – the rigorous way
4. Illustration on lattice based approach
5. cMPS to the rescue?
6. relativistic cMPS and preliminary results

Intuitive definition: canonical quantization



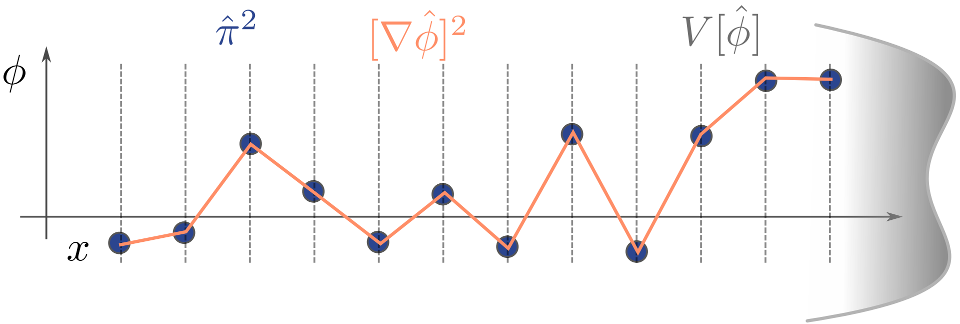
Hamiltonian

A continuum of nearest neighbor coupled anharmonic oscillators

$$\hat{H} = \int_{\mathbb{R}^d} \mathrm{d}^d x \;\; \underbrace{\frac{\hat{\pi}(x)^2}{2}}_{\text{on-site inertia}} + \underbrace{\frac{[\nabla\hat{\phi}(x)]^2}{2}}_{\text{spatial stiffness}} + \underbrace{V(\hat{\phi}(x))}_{\text{on-site potential}}$$

with canonical commutation relations $[\hat{\phi}(x), \hat{\pi}(y)] = i\delta^d(x-y)\mathbb{1}$ (*i.e.* bosons)

Intuitive definition



Hilbert space

Fock space $\mathscr{H}_{\text{QFT}} = \mathscr{F}[L^2(\mathbb{R}^d)]$ – just like $x, p \to (a, a^\dagger)$ do $\hat{\pi}, \hat{\phi} \to \hat{\psi}, \hat{\psi}^\dagger$

$$|\Psi\rangle = \sum_{n=0}^{+\infty} \int \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \underbrace{\varphi_n(x_1, x_2, \cdots, x_n)}_{\text{wave function}} \underbrace{\hat{\psi}^\dagger(x_1)\hat{\psi}^\dagger(x_2)\cdots\hat{\psi}^\dagger(x_n)}_{\text{local oscillator creation}} |\text{vac}\rangle$$

Intuitive definition: functional integral

Insert $\mathbb{1} = \int \mathcal{D}\phi \, |\phi\rangle\langle\phi|$ in expression for correlation functions and $t = i\tau$ gives

Functional integral representation

Representation of correlation functions in terms of random fields

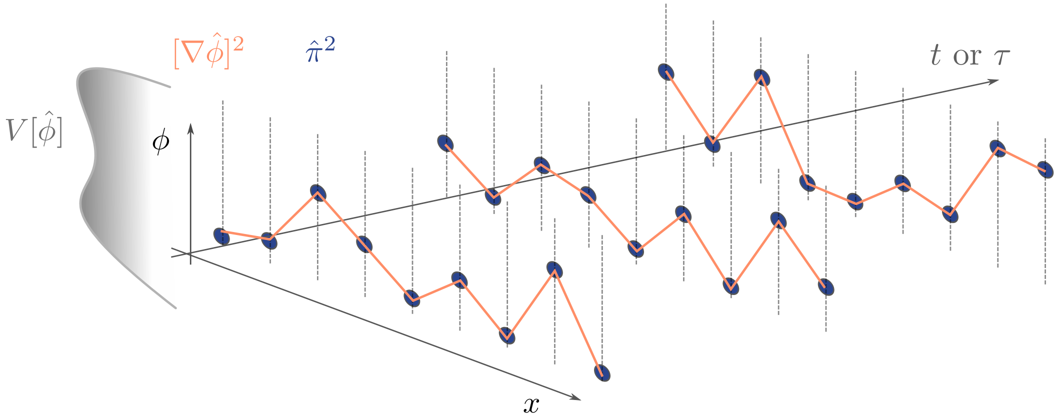
$$\langle 0|\hat{\phi}(\tau_1, x_1)\cdots\hat{\phi}(\tau_n, x_n)|0\rangle := \int \phi(\tau_1, x_1)\cdots\phi(\tau_1, x_n)\, e^{-S(\phi)} \underbrace{\mathcal{D}\phi}_{\text{“Lebesgue measure”}}$$

with the action / weight where $\hat{\pi} \to \frac{\mathrm{d}\phi}{\mathrm{d}\tau}$

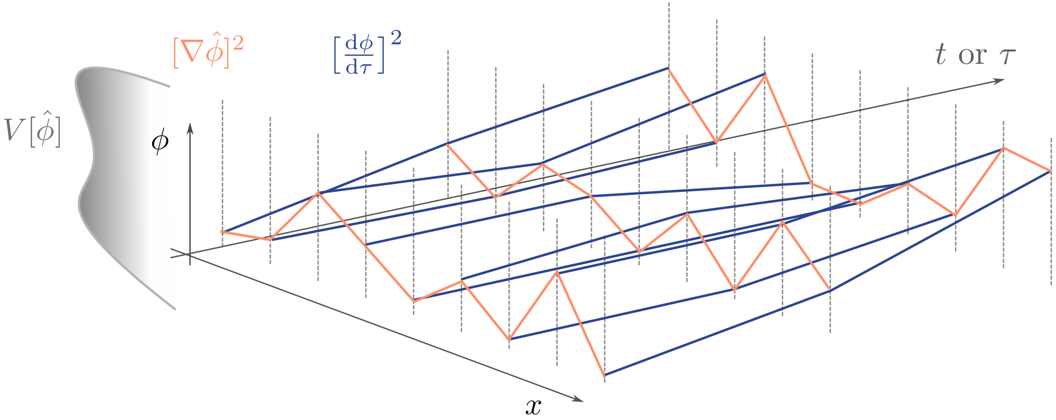
$$S(\phi) = \int \mathrm{d}^d x \, \mathrm{d}\tau \quad \underbrace{\frac{1}{2}\left[\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right]^2}_{\text{inertia a.k.a time stiffness}} + \underbrace{\frac{[\nabla\phi]^2}{2}}_{\text{spatial stiffness}} + \underbrace{V(\phi)}_{\text{on-site potential}}$$

Inertia = time stiffness $\implies$ Euclidean rotation invariance $\implies$ Lorentz

Intuitive definition: functional integral



Intuitive definition: functional integral



What are the problems - Hilbert space approach

The Hamiltonian is ill defined on all states in the Hilbert space because of infinite zero point energy *i.e.* terms $\propto \hat{\psi}(x)\hat{\psi}^\dagger(x)$

$$\langle\Psi_1|\hat{H}|\Psi_2\rangle = \pm\infty \quad \text{and even} \quad \langle \text{vac}|\hat{H}|\text{vac}\rangle \propto \delta^d(0) = +\infty$$

If the divergent vacuum terms are removed, the Hamiltonian is not bounded from below

$$\forall\ |\Psi\rangle \in \mathscr{H},\ \langle\Psi|\hat{H}_{\text{finite}}|\Psi\rangle = \text{finite but} \quad \exists\ \Psi_n \text{ s.t. } \lim_{n\to+\infty}\langle\Psi_n|H_{\text{finite}}|\Psi_n\rangle = -\infty$$

and worse

$$|0\rangle := \lim_{n\to+\infty}|\Psi_n\rangle \notin \mathscr{H}$$

What are the problems - Functional integral approach

Many issues, related to the fact that there is no Lebesgue measure $\mathcal{D}\phi$ on functions [definition issue], and *no equivalent* for $d \geqslant 2$ [real world issue]

The field is not even a function

Entropy dominates energy

$$\langle \phi(x)^2 \rangle = \int \mathrm{d}^d p \, \frac{1}{m^2 + p^2} = +\infty \quad \text{if } d \geqslant 2$$

We penalize irregular and large ϕ, yet the only ones that "typically" occur are so irregular and large the penalty term is ill defined.

How are they are solved in the free case - Hamiltonian

Bogoliubov transform

Go from $\hat{\psi}(x), \hat{\psi}^\dagger(x)$ to $a(p), a^\dagger(p)$ with

$$a(p) = \frac{1}{\sqrt{2}}\left(\sqrt{\omega_p}\,\hat{\phi}(p) + \frac{\hat{\pi}(p)}{\sqrt{\omega_p}}\right) \quad \text{with} \quad \omega_p = \sqrt{p^2 + m^2}$$

which yields

$$H = \int \mathrm{d}p\,\omega_p\,\frac{1}{2}\left(a_p^\dagger a_p + a_p a_p^\dagger\right)$$

Solution

- Take $H_{\mathrm{QFT}} \equiv \,:H:_a$
- $|\text{ground state}\rangle = |\text{vacuum}\rangle_a$
- $\mathscr{H}$ built from $a_{p_1}^\dagger \cdots a_{p_n}^\dagger |\text{vacuum}\rangle_a$

This solves the problematic free part exactly, and allows to define a finite interaction

Rigorous operator definition of ϕ_2^4

Renormalized ϕ_2^4 theory:

$$H = \int \mathrm{dx}\, \frac{:\pi^2:_a}{2} + \frac{:(\nabla\phi)^2:_a}{2} + \frac{m^2}{2} :\phi^2:_a + g:\phi^4:_a$$

note that $:\Diamond:_a$ depends on m!

1. Rigorously defined relativistic QFT without cutoff
2. Vacuum energy density finite
3. Very difficult to solve unless $g \ll m^2$ (perturbation theory)
4. Phase transition around $f_c = \frac{g}{4m^2} \sim 11$

Ways to solve ϕ^4_2

With a lattice of size a (UV cutoff) and fixed number of sites N (IR cutoff)

- Monte-Carlo
- Tensor network renormalization

With a lattice of size a (UV cutoff) and no IR cutoff

- Uniform MPS

With continuous space, an energy cutoff Λ (UV) and an IR cutoff

- Hamiltonian truncation

Without cutoff

- Perturbation theory + Borel-Padé resummation

Lattice ϕ_2^4

Discretize the action:

$$S(\phi) = \sum_{\langle i,j \rangle} \underbrace{\frac{(\phi_i - \phi_j)^2}{2a^2}}_{(\nabla\phi)^2/2} a^2 + \sum_i \frac{1}{2}\mu_a^2 \phi_i^2 + \frac{1}{4}\lambda_a \phi_i^4$$

Taking the limit

The right way to get the continuum limit is to take:

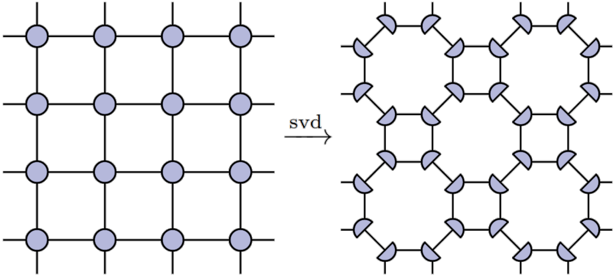
$$\mu_a = \mu a^2 + \frac{3}{2}\log(a) a^2 \lambda$$
$$\lambda_a = \lambda a^2$$

which is equivalent to normal ordering

Basically, at first order in perturbation theory, the ϕ^4 term behaves like a ϕ^2 term times a log divergent constant.

Example with tensor network renormalization

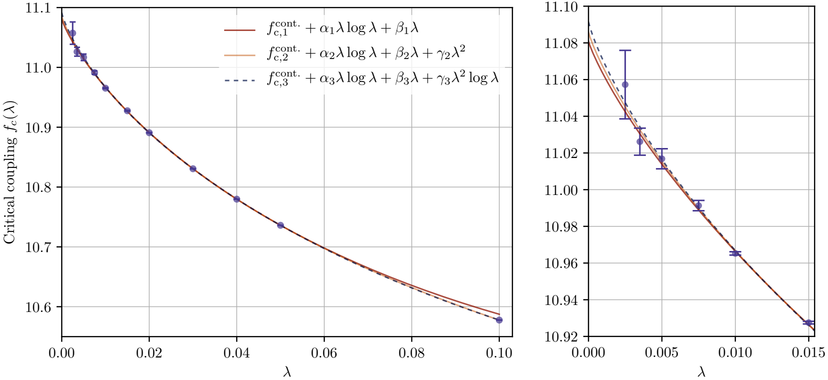
Done with Clément [late 2019 – early 2020]
Discretize ϕ, write $Z = \sum S(\phi)$ as a tensor network and contract it



Technically: UV cutoff (lattice) and IR cutoff (number of RG steps)

Example with tensor network renormalization

Continuum limit taken **numerically**



More costly as the UV cutoff gets small because:

1. Field becomes unbounded at short distance $\rightarrow$ large starting bond dimension

2. More RG steps (with max χ) to get to the same scale

Limitation of numerical continuum limit

Is it a problem of local basis choice?

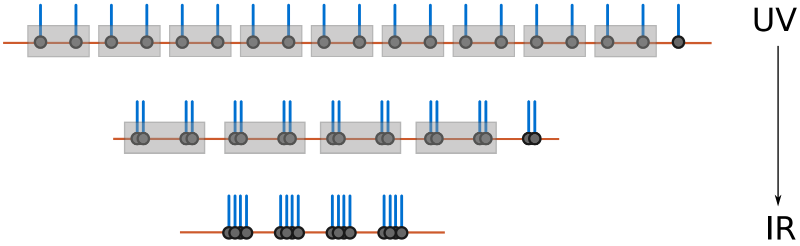
No:

1. UV fixed point is a free CFT, so technically continuum of singular values
2. Interaction is super renormalizable / strongly relevant, so its impact on the tensors $\rightarrow 0$ in continuum limit

$\Longrightarrow$ even theory independent, would apply to QCD, but worse for super-renormalizable theories

Continuous Matrix Product States (cMPS)

Taking the continuum limit of a MPS



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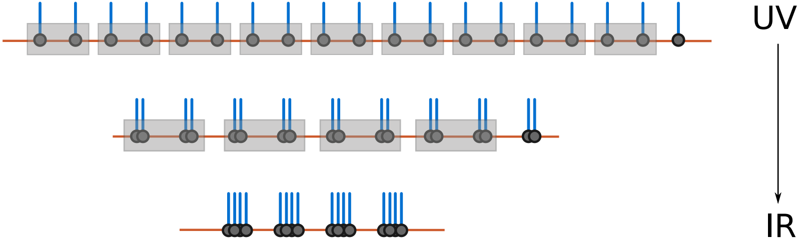
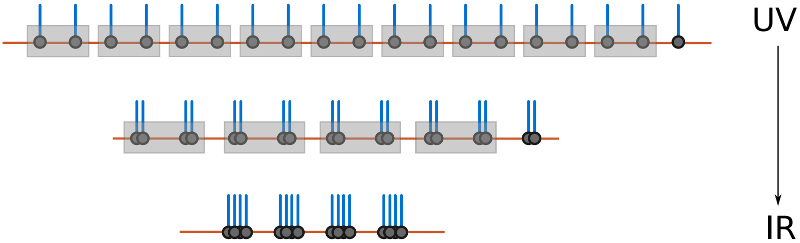


- the bond dimension D stays fixed

Continuous Matrix Product States (cMPS)

Taking the continuum limit of a MPS



- the bond dimension D stays fixed
- the local physical dimension explodes $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \longrightarrow \mathscr{F}(L^2([x, x+dx]))$.

$\Longrightarrow$ **Spins** become **fields** – ($\simeq$ central limit theorem $\simeq$)

Continuous Matrix Product States

Type of ansatz for bosons on a fine grained $d = 1$ lattice

- Matrices $A_{i_k}(x)$ where the index i_k corresponds to $\psi^{\dagger i_k}(x)|0\rangle$ in physical space.

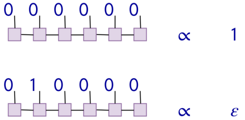
Informal cMPS definition

$$A_0 = \mathbb{1} + \varepsilon Q$$
$$A_1 = \varepsilon R$$
$$A_2 = \frac{(\varepsilon R)^2}{\sqrt{2}}$$
$$\dots$$
$$A_n = \frac{(\varepsilon R)^n}{\sqrt{n}}$$

so we go from ∞ to 2 matrices

Fixed by:

- Finite particle number



- Consistency

Continuous Matrix Product States

Definition

$$|Q, R, \omega\rangle = \langle\omega_L|\mathcal{P}\exp\left\{\int_0^L dx\ Q \otimes \mathbb{1} + R \otimes \psi^\dagger(x)\right\}|\omega_R\rangle\ |0\rangle_\psi$$

- Q, R are $D \times D$ matrices,
- $|\omega_L\rangle$ and $|\omega_R\rangle$ are boundary vectors $\in \mathbb{C}^D$, for p.b.c. $\langle\omega_L| \cdot |\omega_R\rangle \to \mathrm{tr}[\,\cdot\,]$
- $[\psi(x), \psi^\dagger(y)] = \delta(x - y)$

Idea:

Continuous Matrix Product States

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- $[\psi(x), \psi^\dagger(y)] = \delta(x - y)$

Idea:

$$\begin{aligned} A(x) &\simeq A_0 \mathbb{1} + A_1 \psi^\dagger(x) \\ &\simeq \mathbb{1} \otimes \mathbb{1} + \varepsilon Q \otimes \mathbb{1} + \varepsilon R \otimes \psi^\dagger(x) \\ &\simeq \exp \left[\varepsilon \left(Q \otimes \mathbb{1} + R \otimes \psi^\dagger(x) \right) \right] \end{aligned}$$

Computations

Some correlation functions

$$\left\langle \hat{\psi}(x)^\dagger \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{TL}(R \otimes \overline{R})\right]$$

$$\left\langle \hat{\psi}(x)^\dagger \hat{\psi}(0)^\dagger \hat{\psi}(0) \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{T(L-x)}(R \otimes \overline{R}) e^{Tx}(R \otimes \overline{R})\right]$$

$$\left\langle \hat{\psi}(x)^\dagger \left[-\frac{d^2}{dx^2}\right] \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{TL}([Q, R] \otimes [\overline{Q}, \overline{R}])\right]$$

with $T = Q \otimes \mathbb{1} + \mathbb{1} \otimes \bar{Q} + R \otimes \bar{R}$

Example

Lieb-Liniger Hamiltonian

$$\mathcal{H} = \int_{-\infty}^{+\infty} dx \left[\frac{d\hat{\psi}^\dagger}{dx}\frac{d\hat{\psi}}{dx} - \mu \hat{\psi}^\dagger \hat{\psi} + c \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi}\right]$$

Solve by **minimizing**: $\langle Q, R | \mathcal{H} | Q, R \rangle = f(Q, R)$

Standard CMPS and ϕ^4

Applying cMPS to the ϕ^4 Hamiltonian

$$\langle Q, R|\hat{h}_{\phi^4}|Q, R\rangle = \infty$$

Oh no!

The short distance behavior of cMPS is the wrong one, even the free theory is hard to approximate.

Towards relativistic CMPS

Local basis in position of the QFT: $\psi^\dagger$, ϕ, π, $|0\rangle_\psi$

Diagonal basis of the free part: $a_k^\dagger$, $|0\rangle_a$

Bogoliubov transform

Go from $\hat{\psi}(x), \hat{\psi}^\dagger(x)$ to $a(p), a^\dagger(p)$ with

$$a(p) = \frac{1}{\sqrt{2}}\left(\sqrt{\omega_p}\,\hat{\phi}(p) + \frac{\hat{\pi}(p)}{\sqrt{\omega_p}}\right) \quad \text{with} \quad \omega_p = \sqrt{p^2 + m^2}$$

which yields

$$H_0 = \int \mathrm{d}p\, \omega_p\, \frac{1}{2}\left(a_p^\dagger a_p + a_p a_p^\dagger\right)$$

Go from $|0\rangle_\psi$ to $|0\rangle_a$

and

Go from $\psi(x)$ to $a(x) = \int \mathrm{d}p\; a(p)\mathrm{e}^{ipx} \neq \psi(x)$

Relativistic CMPS

Definition

$$|R, Q\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q \otimes \mathbb{1} + R \otimes a^{\dagger}(x)\right]\right\}|0\rangle_a$$

Some properties

1. $|0, 0\rangle = |0\rangle_a$ is the ground state of H_0 hence exact CFT UV fixed point
2. $\langle Q, R|h_{\phi^4}|Q, R\rangle$ is finite for all Q, R (not trivial)

Consequence on the Hamiltonian

H is local in $\psi(x)$, not in $a(x)$...

$$
\begin{aligned}
H = &\int \mathrm{d}x_1 \mathrm{d}x_2 D(x_1 - x_2) a^\dagger(x_1) a(x_2) \\
&+ \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 K(x_1, x_2, x_3, x_4) a(x_1) a(x_2) a(x_3) a(x_4) + 4 a^\dagger aaa + 3 a^\dagger a^\dagger aa \\
&+ \mathrm{h.c.}
\end{aligned}
$$

But exponentially decreasing: K is horrible, but decays $\propto e^{-m|x|}$.

The nightmarish optimization

Compute $e_0 = \langle Q, R | h_{\phi^4} | Q, R \rangle$ and $\nabla_{Q,R} e_0$

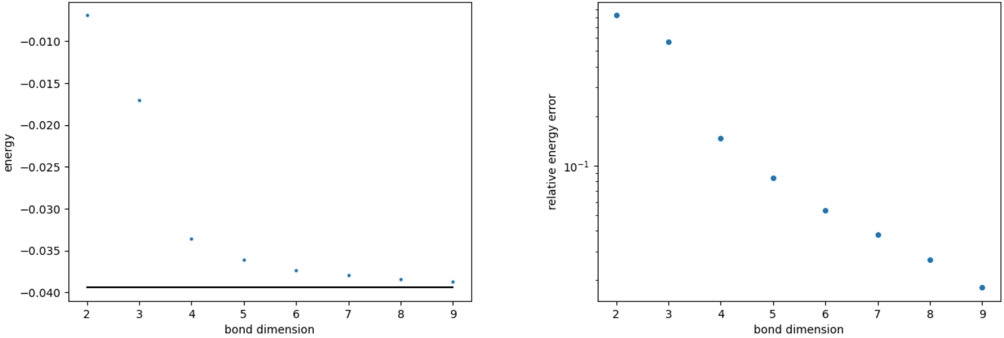
1. Contains an algebraic part identical to standard cMPS
2. Involves horrible quadruple integrals without analytic solutions

Optimization with naive gradient descent, BFGS, or conjugate gradient leads to plateaus $\implies$ does not work

One needs to do TDVP with a metric, slightly more complicated but works.

Preliminary results

After a scary amount of optimization, test at $g = 1$ (deeply non perturbative)



Seems exponentially convergent! First rigorous bound on ϕ^4 energy

What now

1. Still very costly (3 days, 40 cores for $D = 9$)
2. Can get modest improvement by changing the a_m to $a_{\tilde{m}}$ for different masses (in progress)
3. Need to compute gap with TDVP
4. Get closer to criticality

Summary

1. New ansatz for $1+1$ relativistic QFT
2. No cutoff, UV or IR (a first?)
3. UV is captured exactly even at $D = 0$
4. Rigorous (variational)
5. Gives some hints to improve numerical continuous limits (e.g. in lattice QCD)