Tensor Networks for Quantum Field Theory

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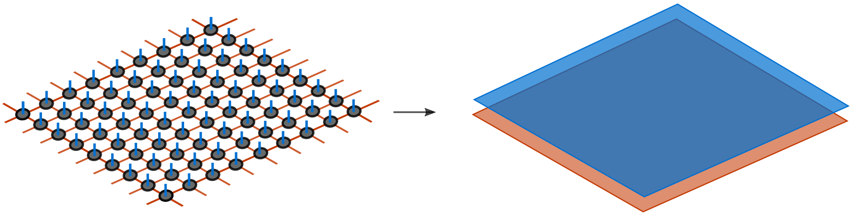


Universität Leipzig
via Zoom, from Munich, Germany
December 18th, 2020

Alexander von Humboldt
Stiftung/Foundation

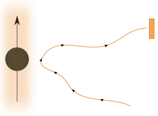
MPQ

My profile

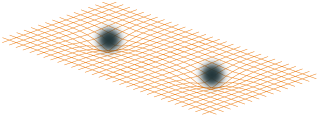
3 years PhD (ENS), **4** years postdoc MPQ

3 main lines of inquiry

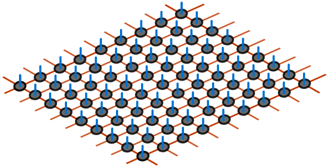
Continuous measurement
How to gently measure and control quantum systems?



Gravity and quantum
Could gravity, in principle, not be quantum?



Many-body & QFT
How to efficiently parameterize many-body and QFT states



Quantum field theory: a bit of philosophy

Two ways to attack *real world* quantum field theories non-perturbatively

1. Start **simpler** so that it becomes **simpler** [*e.g.* self interacting scalar field ϕ_2^4]

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ϕ_2^4 - pile of dirt

QCD - Everest

$\mathcal{N} = 4$ *SYM* - Chrysler building

2. is a good way to make fast progress first, but is limited in what it can achieve

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Goal - ideal - philosophy: an apology of the pile of dirt approach

Abandon analytical solutions, but find robust methods that can solve simple QFTs non-perturbatively and, if possible, to machine precision, *without cheating*.

more on this on `tilloy.wordpress.com`

QFT: simplest non-trivial example

ϕ_2^4 is a relativistic QFT in $d = 1+1$ dimensions defined by the Hamiltonian:

$$H = \int \mathrm{d}x \, \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}(\nabla\hat{\phi})^2 + \frac{m^2}{2}\hat{\phi}^2 + \frac{g}{4} :\hat{\phi}^4:$$

- It exists as a true QFT (rigorously defined by constructive field theorists)
- It is not integrable, has no special structure simplifying computations
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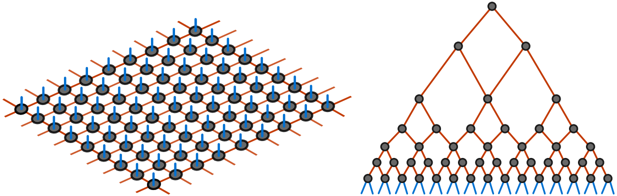
S. Rychkov challenge $\simeq$ 2015

Compute everything one could want to compute about this model for all values of g. In particular find the position of the phase transition (approximately at $g/m^2 \simeq 10 \pm 20\%$)

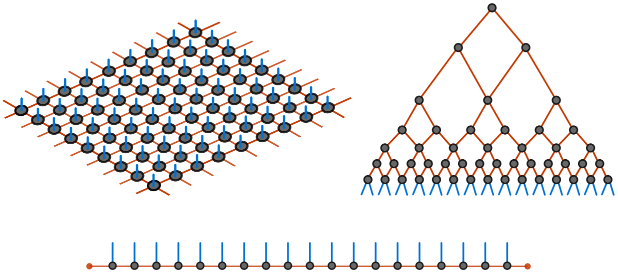
Tensor network states: a tool



Applications

- Quantum information theory
- Statistical Mechanics
- Quantum gravity
- **Many-body quantum**

Tensor network states: a tool



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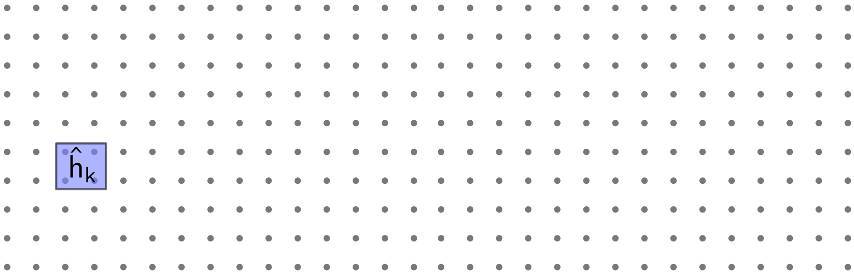
Negative theology

- **Not** covariant/geometric objects $g_{\mu\nu}$ or $R^{\sigma}_{\mu\nu\kappa}$
- **Not** tensor **models** [Rivasseau, Gurau, ...]

Outline

1. Tensor networks on the lattice
2. Bringing QFT to the lattice
3. Bringing tensor networks to the continuum

Many-body problem



Problem

Finding low energy states of

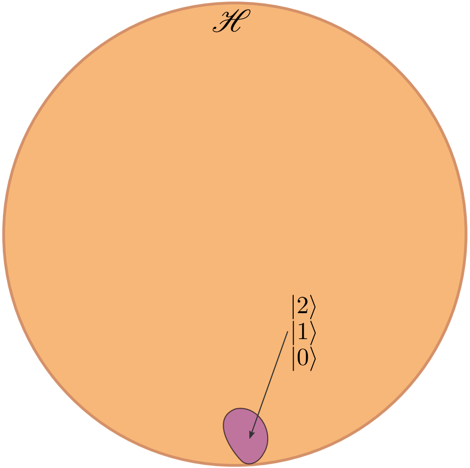
$$\hat{H} = \sum_{k=1}^{N} \hat{h}_k$$

is **hard** because $\dim \mathscr{H} \propto D^N$

Possible solutions

- Perturbation theory
- Monte Carlo
- Bootstrap IR fixed point
- **Variational optimization** (*e.g.* Mean Field, Hamiltonian truncation, tensor networks)

Variational optimization



Generic state $\in \mathscr{H}$:

$$|\psi\rangle = \sum_{i_1, i_2, \cdots, i_n}^{D} c_{i_1, i_2, \cdots, i_N} |i_1, \cdots, i_N\rangle$$

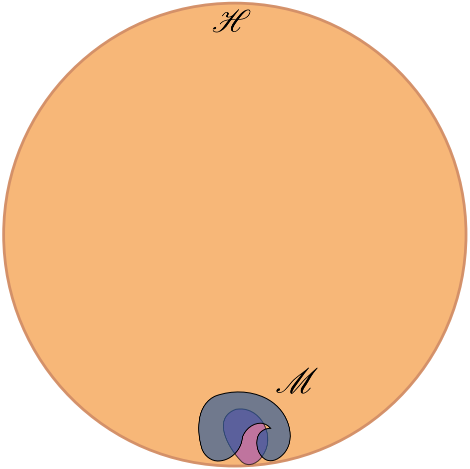
Exact variational optimization

To find the ground state:

$$|0\rangle = \min_{|\psi\rangle \in \mathscr{H}} \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

- $\dim \mathscr{H} = D^N$

Variational optimization



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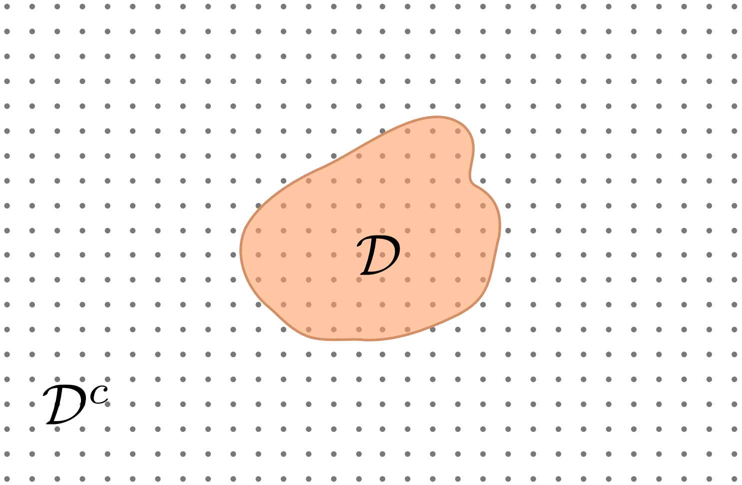
Approx. variational optimization

To find the ground state:

$$|0\rangle = \min_{|\psi\rangle \in \mathscr{M}} \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

- $\dim \mathscr{M} \propto \mathrm{Poly}(N)$ or fixed

Interesting states are weakly entangled



Low energy state

$|\psi\rangle = |0\rangle$ or $|1\rangle$ ...

Reduced density matrix

$\rho = \mathrm{tr}_{\mathcal{D}^c}\big[|\psi\rangle\langle\psi|\big]$

Entanglement entropy

$S = -\mathrm{tr}\big[\rho \log \rho\big]$

Area law

$$S \propto |\partial \mathcal{D}|$$

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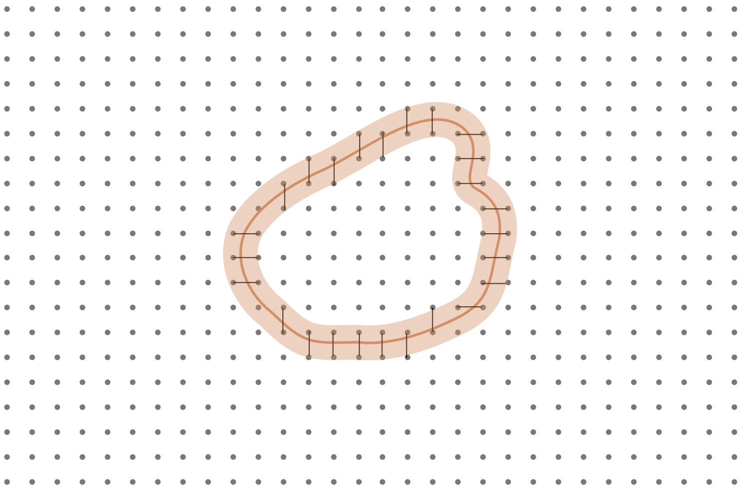

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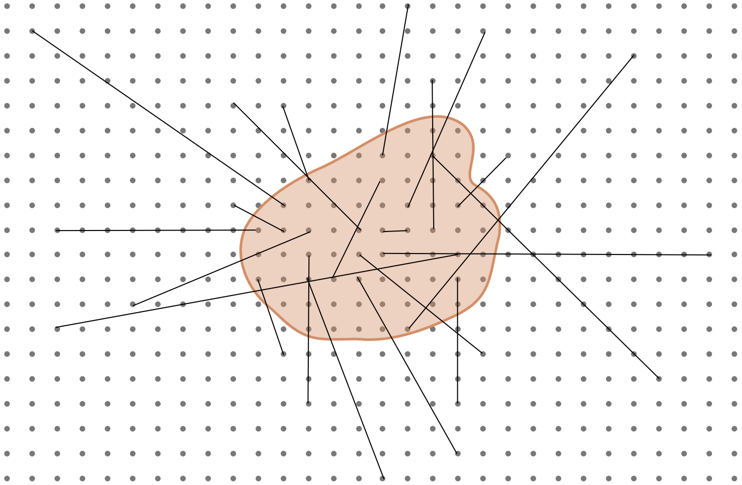
Typical states are strongly entangled



Random state

$|\psi\rangle = U_{\text{Haar}}|\text{trivial}\rangle$

Reduced density matrix

$\rho = \text{tr}_{\mathcal{D}^c}\left[|\psi\rangle\langle\psi|\right]$

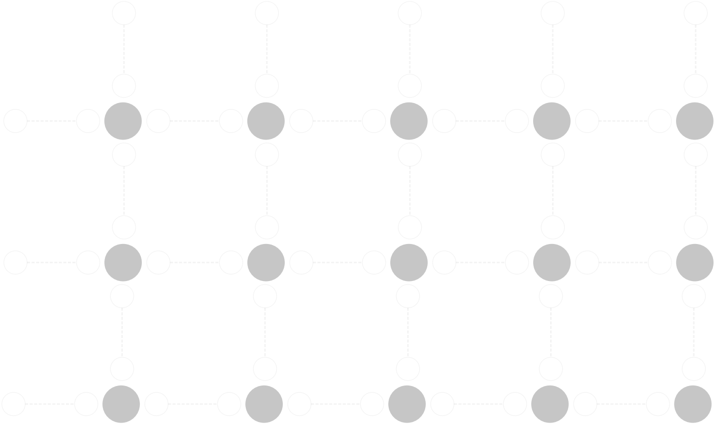
Entanglement entropy

$S = -\text{tr}\left[\rho \log \rho\right]$

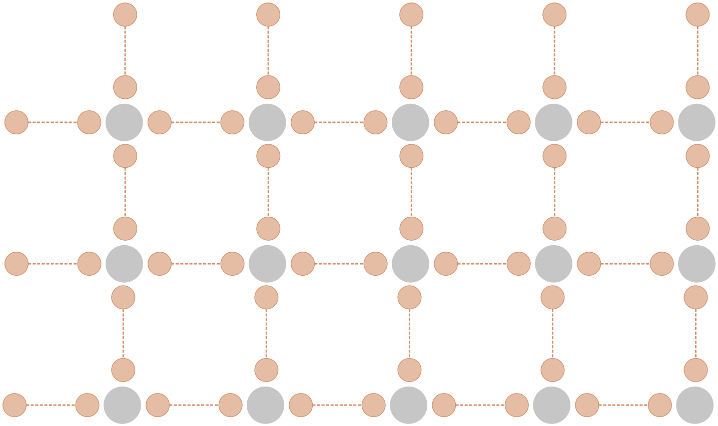
Volume law

$$S \propto |\mathcal{D}|$$

Constructing weakly entangled states



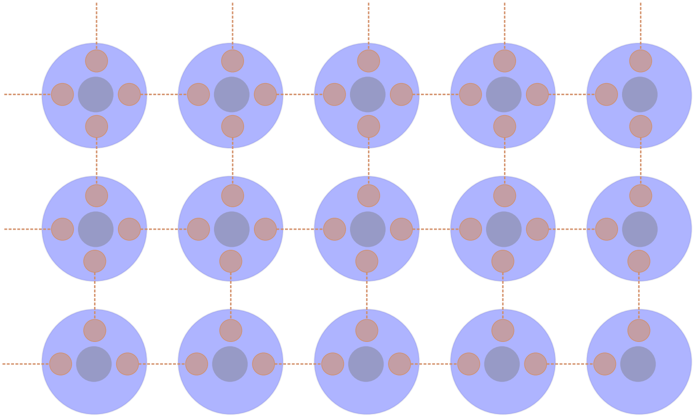
Constructing weakly entangled states



1. Put auxiliary **maximally entangled** states between sites

$$= \sum_{j=1}^{\chi} |j\rangle|j\rangle$$

Constructing weakly entangled states



1. Put auxiliary **maximally entangled** states between sites

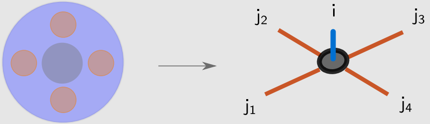
 $= \sum_{j=1}^{\chi} |j\rangle|j\rangle$

2. Map to initial Hilbert space on each site

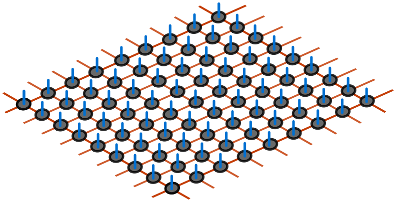
 $= A : \mathbb{C}^{4\chi} \to \mathbb{C}^{D}$

Tensor network states: definition

Why "tensor" network?



$A : \mathbb{C}^{4X} \to \mathbb{C}^{D} \quad \longrightarrow \quad A^{i}_{j_1, j_2, j_3, j_4}$

$|A\rangle =$ 

with tensor contractions on links

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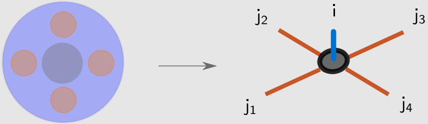


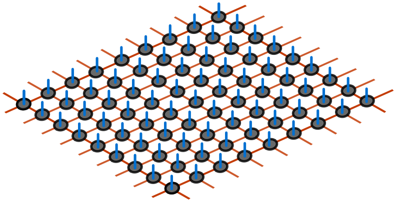
$A : \mathbb{C}^{4\chi} \to \mathbb{C}^{D} \quad \longrightarrow \quad A^{i}_{j_1,j_2,j_3,j_4}$

Optimization

Find best A for fixed χ $\quad (D \times \chi^4$ coeff.)

$$E_0 \simeq \min_A \frac{\langle A|\widehat{H}|A\rangle}{\langle A|A\rangle}$$

for example go down $\frac{\partial E}{\partial A^{i}_{j_1,j_2,j_3,j_4}}$

$|A\rangle =$ 

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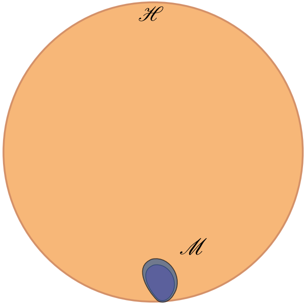
Some facts

$d = 1$ spatial dimension



Theorems (colloquially)

1. For gapped H, tensor network states $|A\rangle$ approximate well $|0\rangle$ with χ fixed
2. **All** $|A\rangle$ are ground states of gapped H

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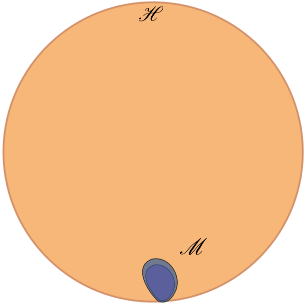
$d = 1$ spatial dimension



Theorems (colloquially)

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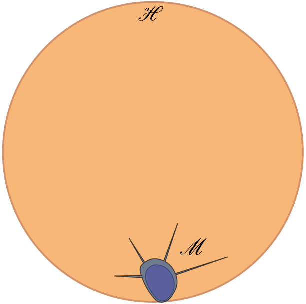
$d \geqslant 2$ spatial dimension



Folklore

1. For gapped H, tensor network states $|A\rangle$ approximate well $|0\rangle$ with χ fixed
2. **Most** $|A\rangle$ are ground states of gapped H

From condensed matter to QFT

Tensor network are excellent **theoretically** and **numerically** but limited to the **lattice**

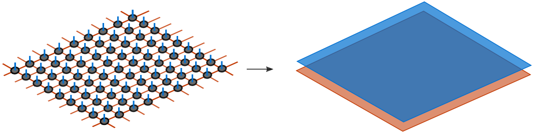
2 options:

- Discretize QFT, solve with best known tensor network algorithms

$$S(\phi) = \sum_{\langle i,j \rangle} \underbrace{\frac{(\phi_i - \phi_j)^2}{2a^2}}_{(\nabla\phi)^2/2} a^2 + \sum_i \frac{1}{2}\mu_a^2 \phi_i^2 + \frac{1}{4}\lambda_a \phi_i^4$$

  and take $a \to 0$

- Take the continuum limit of tensor networks, and apply to QFT directly



Lattice ϕ^4_2

Discretize the action:

$$S(\phi) = \sum_{\langle i,j \rangle} \underbrace{\frac{(\phi_i - \phi_j)^2}{2a^2}}_{(\nabla\phi)^2/2} a^2 + \sum_i \frac{1}{2}\mu_a^2 \phi_i^2 + \frac{1}{4}\lambda_a \phi_i^4$$

Taking the limit

The right way to get the continuum limit is to take:

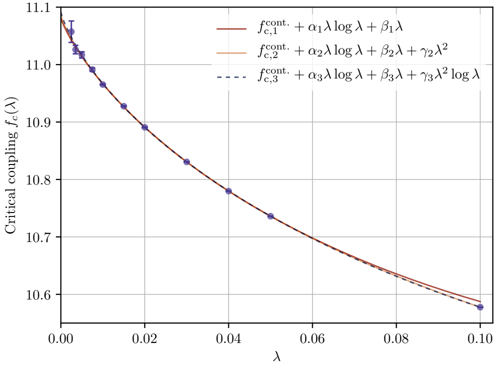
$$\mu_a^2 = \mu^2 a^2 + \frac{3}{2}\log(a) a^2 \lambda$$
$$\lambda_a = \lambda a^2$$

which is equivalent to normal ordering

At first order in perturbation theory, the ϕ^4 term behaves like a ϕ^2 term times a log divergent constant.

Results with GILT tensor renormalization

With C. Delcamp, we found the critical point $f_c = \lim_{a \to 0} \frac{\lambda_a}{\mu_a^2}$ in the continuum limit to the highest precision ever `arXiv:2003.12993`



Method	$f_c^{\text{cont.}}$	Year	Ref.
Tensor network coarse-graining	10.913(56)	2019	[9]
Borel resummation	11.23(14)	2018	[6]
Renormalized Hamil. Trunc.	11.04(12)	2017	[5]
Matrix Product States	11.064(20)	2013	[7]
Monte Carlo	11.055(20)	2019	[15]
This work	11.0861(90)	2020	

TABLE I. Comparison of several estimates of the critical coupling constant $f_c^{\text{cont.}}$ in the continuum obtained using different methods.

The $a \log a$ correction of the critical point position as a function of lattice spacing a was not known before

Directly in the continuum

What was known (since 2010)

Continuous matrix product states for $1+1$ dimensional **non-relativistic** QFT

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What was known (since 2010)

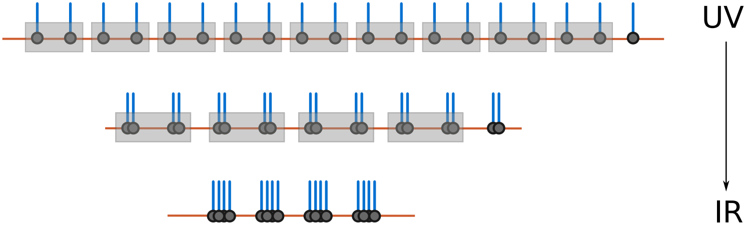
Continuous matrix product states for $1+1$ dimensional **non-relativistic** QFT

My contribution

- Define continuous tensor networks for $1+d$ dimensional **non-relativistic** QFT [with I. Cirac]
- Demonstrate that they have the right UV properties and fast convergence [with T. Karanikolaou]
- Define relativistic continuous matrix product states for $1+1$ **relativistic** QFT [preliminary numerics]

Continuous Matrix Product states

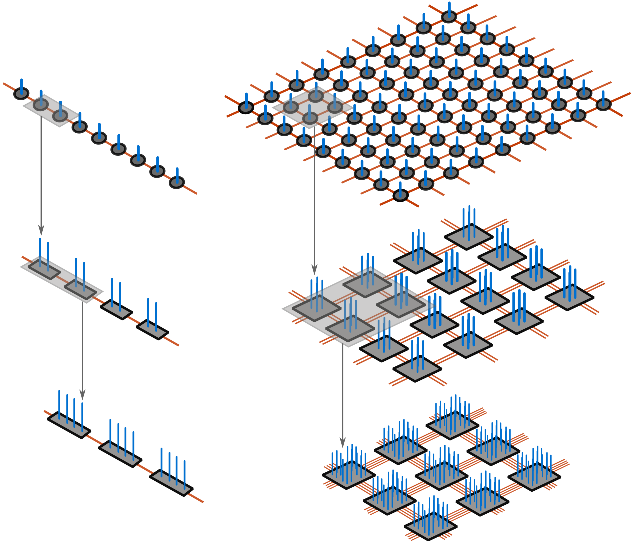
[Verstraete & Cirac 2010]: continuum limit of **M**atrix **P**roduct **S**tates ($d = 1$ tensor networks)



Works for Lieb-Liniger model (boson with contact interactions)
Best method on the market for $1 + 1$ **non-relativistic** QFT

But no version for $d + 1$ QFT, even “no-go” theorems

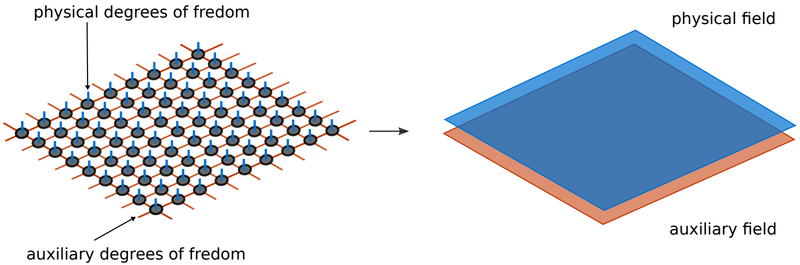
Continuous Tensor Networks: blocking



Upon **blocking**:

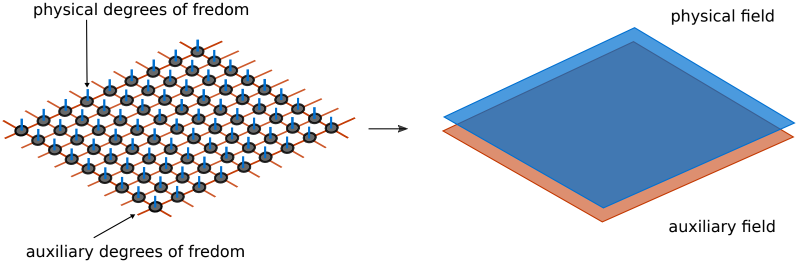
- The **physical** Hilbert space dimension D increases
- The **bond** (auxiliary space) dimension χ increases too

Result



AT, J. I. Cirac, *Phys. Rev. X 2019*

Result



AT, J. I. Cirac, *Phys. Rev. X 2019*

Continuous tensor network state (heuristically)

State $|\alpha\rangle$ of $d+1$ QFT from an auxiliary d dimensional theory of random fields ϕ:

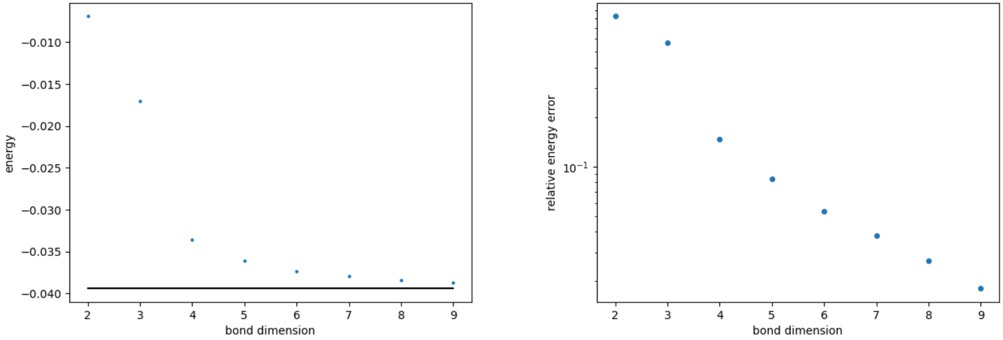
$$|\alpha\rangle = \int \mathcal{D}\phi \, \exp\left\{ -\int \mathrm{d}^d x \, \mathcal{L}[\phi(x)] - \alpha[\phi(x)] \underbrace{\hat{\psi}^\dagger(x)}_{\text{creation}} \right\} |\Omega\rangle$$

1. Genuine continuum limit of discrete tensor networks
2. Right UV scaling and exponential convergence to the ground state as the number of auxiliary fields ϕ in increased arXiv:2006.13143

Preliminary continuous relativistic results in $1+1$

$$H = \int \mathrm{d}x \, \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}(\nabla\hat{\phi})^2 + \frac{m^2}{2}\hat{\phi}^2 + \frac{g}{4} :\hat{\phi}^4:$$

Test of a brand new **relativistic continuous matrix product state ansatz** at $g = 4$ (deeply non perturbative). No **UV** nor **IR** cutoffs!



Seems exponentially convergent! First rigorous bound on ϕ^4 energy

Summary of tensor networks in QFT

Tensor networks are promising for non-perturbative QFT:

- They are already the best numerical method for QFT in $1+1$ dimensions
- They can now be applied to (non-relativistic) QFT in $1+d$
- They will very soon give **rigorous** results for relativistic QFT in $1+1$ dimensions

In the near future:

- Push lattice based approach to lattice gauge theory (go beyond scalar)
- Push continuous approach to relativistic $1+d$