Variational method in relativistic QFT

without cutoff

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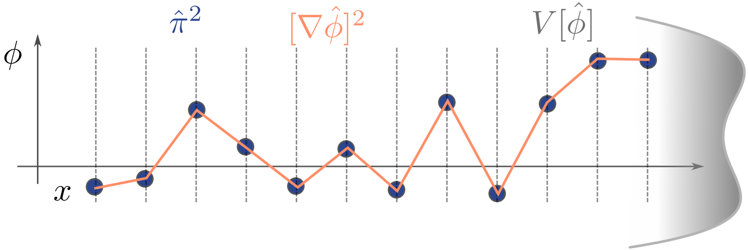

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Quantum field theory: a bit of philosophy

Two ways to attack *real world* quantum field theories non-perturbatively

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ϕ_2^4 - pile of dirt

QCD - Everest

$\mathcal{N} = 4$ *SYM* - Chrysler building

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Goal - ideal - philosophy: an apology of the pile of dirt approach

Abandon analytical solutions, but find robust methods that can solve simple QFTs non-perturbatively and, if possible, to machine precision, *without cheating*.

more on this on `tilloy.wordpress.com`

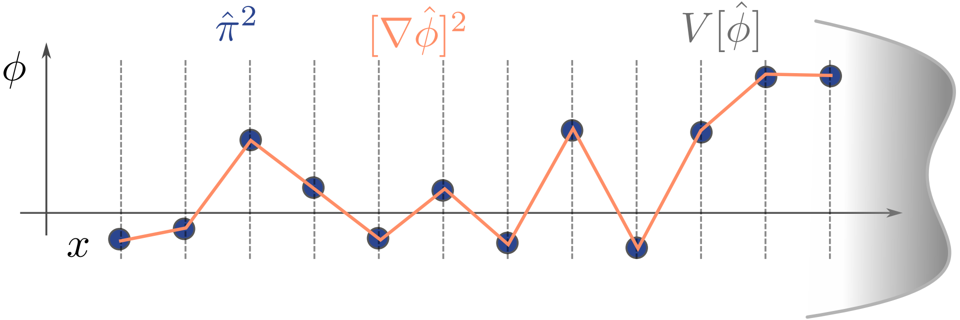
Outline

1. ϕ_2^4 for beginners
2. The variational method
3. Tensor networks on the lattice
4. Matrix product states and their continuum limit
5. Going relativistic
6. Results and discussion

ϕ_2^4 for beginners

and condensed matter theorists

Intuitive definition: canonical quantization



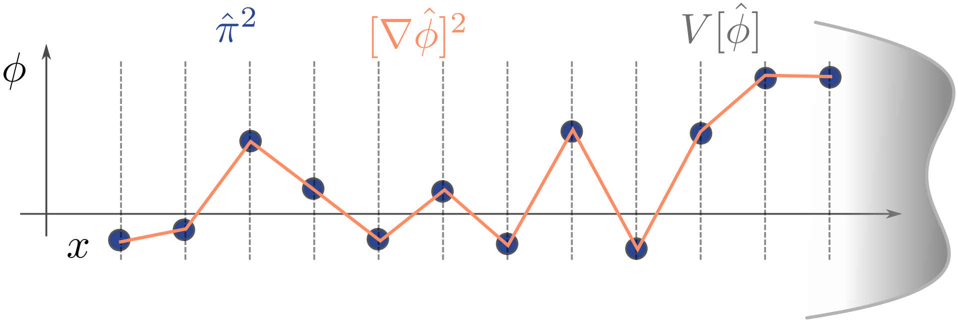
Hamiltonian

A continuum of nearest neighbor coupled anharmonic oscillators

$$\hat{H} = \int_{\mathbb{R}^d} \mathrm{d}^d x \; \underbrace{\frac{\hat{\pi}(x)^2}{2}}_{\text{on-site inertia}} + \underbrace{\frac{[\nabla\hat{\phi}(x)]^2}{2}}_{\text{spatial stiffness}} + \underbrace{V(\hat{\phi}(x))}_{\text{on-site potential}}$$

with canonical commutation relations $[\hat{\phi}(x), \hat{\pi}(y)] = i\delta^d(x-y)\mathbb{1}$ (*i.e.* bosons)

Intuitive definition



Hilbert space

Fock space $\mathscr{H}_{\text{QFT}} = \mathscr{F}[L^2(\mathbb{R}^d)]$ – just like $x, p \to (a, a^\dagger)$ do $\hat{\pi}, \hat{\phi} \to \hat{\psi}, \hat{\psi}^\dagger$

$$|\Psi\rangle = \sum_{n=0}^{+\infty} \int \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \underbrace{\varphi_n(x_1, x_2, \cdots, x_n)}_{\text{wave function}} \underbrace{\hat{\psi}^\dagger(x_1)\hat{\psi}^\dagger(x_2)\cdots\hat{\psi}^\dagger(x_n)}_{\text{local oscillator creation}} |\text{vac}\rangle$$

What are the problems - Hilbert space approach

The Hamiltonian is ill defined on all states in the Hilbert space because of infinite zero point energy *i.e.* terms $\propto \widehat{\psi}(x)\widehat{\psi}^\dagger(x)$

$$\langle\Psi_1|\widehat{H}|\Psi_2\rangle = \pm\infty \quad \text{and even} \quad \langle\text{vac}|\widehat{H}|\text{vac}\rangle \propto \delta^d(0) = +\infty$$

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If the divergent vacuum terms are removed, the Hamiltonian is not bounded from below

$$\forall\ |\Psi\rangle \in \mathscr{H},\ \langle\Psi|\widehat{H}_{\text{finite}}|\Psi\rangle = \text{finite but} \quad \exists\ \Psi_n \text{ s.t.} \lim_{n\to+\infty}\langle\Psi_n|H_{\text{finite}}|\Psi_n\rangle = -\infty$$

How are they are solved in the free case - Hamiltonian

Bogoliubov transform

Go from $\hat{\psi}(x), \hat{\psi}^\dagger(x)$ to $a(p), a^\dagger(p)$ with

$$a(p) = \frac{1}{\sqrt{2}}\left(\sqrt{\omega_p}\,\hat{\phi}(p) + \frac{\hat{\pi}(p)}{\sqrt{\omega_p}}\right) \quad \text{with} \quad \omega_p = \sqrt{p^2 + m^2}$$

which yields

$$H_0 = \int \mathrm{d}p\, \omega_p \,\frac{1}{2}\left(a_p^\dagger a_p + a_p a_p^\dagger\right)$$

Solution

- Take $H_{\text{QFT}} \equiv \,:H:_a$
- $|\text{free ground state}\rangle = |\text{vacuum}\rangle_a$
- $\mathscr{H}$ built from $a_{p_1}^\dagger \cdots a_{p_n}^\dagger |\text{vacuum}\rangle_a$

This solves the problematic free part exactly, and allows to define a finite interaction (in $1+1$)

Rigorous operator definition of ϕ_2^4

Renormalized ϕ_2^4 theory

$$H = \int \mathrm{d}x \, \frac{:\pi^2:_a}{2} + \frac{:(\nabla\phi)^2:_a}{2} + \frac{m^2}{2} :\phi^2:_a + g :\phi^4:_a$$

(note that $:\Diamond:_a$ depends on m)

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1. Rigorously defined relativistic QFT without cutoff (Wightman QFT)
2. Vacuum energy density finite
3. Very difficult to solve unless $g \ll m^2$ (perturbation theory)
4. Phase transition around $f_c = \frac{g}{4m^2} = 11$ i.e. $g \simeq 2.7$ in mass units

The variational method

Solving the non-exactly solvable by guessing well

Ways to solve the non-exactly-solvable

The two main games in town

1. Perturbative expansions (+ Borel-Padé resummation)
2. Lattice Monte Carlo

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Two "new" deterministic non-perturbative options:

1. Variational method $\rightarrow$ focus of today
2. Non-perturbative renormalization group (Kadanoff, FRG, Tensor RG, etc.)

The two new methods now rule on (low dimensional) lattice problems thanks to tensor networks $\rightarrow$ QFT?

The variational method

In the Hamiltonian formulation:

- Guess a **finite dimensional submanifold** $\mathscr{M}$ of the QFT Hilbert space $\mathscr{H}$
- Find the ground state by minimizing $\langle H \rangle$:

$$|\text{ground}\rangle \simeq |\psi\rangle = \underset{\mathscr{M}}{\text{argmin}} \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}$$

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Example: naive Hamiltonian truncation

With an IR cutoff, momenta are discrete. Take as submanifold $\mathscr{M}$ the **vector space** spanned by:

$$a^\dagger_{k_1} a^\dagger_{k_2} \cdots a^\dagger_{k_r} |0\rangle_a$$

where $r \leqslant r_{\text{max}}$ and $k \leqslant k_{\text{max}}$ (one possible truncation)

Feynman's objection

Feynman's requirement for variational wavefunctions in RQFT (1987)

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All methods so far break one at least:

- Hamiltonian truncation fails at 1 (but works fairly well through its renormalized refinements)
- Tensor networks succeed at 1 and 2 but fail (a priori) at 3

Haegeman-Cirac-Osborne-Verschelde-Verstraete fix of 2010: regulate the UV by adding a Lagrange multiplier in the Hamiltonian $H \to H + \frac{1}{\Lambda^2}\text{regulator}$

Tensor network states

The best guess for the many-body problem on the lattice

Tensor networks in a nutshell

Tensor network states provide a **compressed** representation of low energy states of local Hamiltonians on the lattice

- Compression possible because **area law**: such states are weakly entangled
- The "bond dimension" D quantifies the number of parameters
- In 1 space dimension, provably efficient (cost poly D, error superpoly $1/D$)

Matrix Product States (MPS)

Definition

A MPS for a translation invariant chain of N qudits ($\mathbb{C}^d$) with periodic boundary conditions is a state

$$|\psi(A)\rangle := \sum_{i_1, i_2, \ldots, i_N} \mathrm{tr}\left[A_{i_1} A_{i_2} \cdots A_{i_N}\right] |i_1, i_2, \ldots, i_N\rangle$$

where A_i are d matrices $\in \mathcal{M}_D(\mathbb{C})$.

- The matrices A_i for $i = 1 \ldots d$ are the free parameters
- The size D of the matrices is the **bond dimension** (quantifies freedom)
- Correlation functions (and $\langle H \rangle$) efficiently computable
- Optimizing over A provably gives good results for gapped H

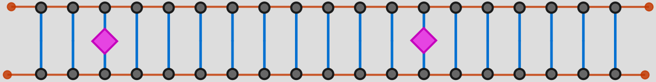
MPS in graphical notation

$|A, L, R\rangle = \sum_{i_1, i_2, \cdots, i_n} \langle \mathrm{L}|A_{i_1}(1) A_{i_2}(2) \cdots A_{i_n}(n)|\mathrm{R}\rangle \; |i_1, \cdots, i_n\rangle$

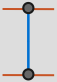
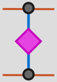
Notation: $[A_i]_{\mathbf{k},\mathbf{l}} =$  and k —— $l = \sum \delta_{k,l}$ gives:

$|A, L, R\rangle =$ 

Example: computation of correlations

$\langle A|\mathcal{O}(i_k)\mathcal{O}(i_\ell)|A\rangle =$ 

can be done efficiently by iterating 2 maps:

$\Phi =$  and $\Phi_{\mathcal{O}} =$ 

(Continuous) matrix product states

Taking the simplest tensor network and scaling it up to QFT

Continuous Matrix Product States

Type of ansatz for bosons on a fine grained lattice

- Matrices $A_{i_k}(x)$ where the index i_k corresponds to $\psi^{\dagger i_k}(x)|0\rangle$ in physical space.

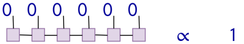
Informal cMPS definition

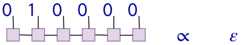
$$A_0 = \mathbb{1} + \varepsilon Q$$
$$A_1 = \varepsilon R$$
$$A_2 = \frac{(\varepsilon R)^2}{\sqrt{2}}$$
$$\dots$$
$$A_n = \frac{(\varepsilon R)^n}{\sqrt{n}}$$

so we go from ∞ to 2 matrices

Fixed by:

- Finite particle number





- Consistency

Continuous Matrix Product States

Introduced by Verstraete and Cirac in 2010

Definition

$$|Q, R, \omega\rangle = \mathrm{tr}\left[\mathcal{P}\exp\left\{\int_0^L dx\ Q \otimes \mathbb{1} + R \otimes \psi^\dagger(x)\right\}\right] |0\rangle_\psi$$

- Q, R are $D \times D$ matrices,
- The trace is taken over this auxiliary matrix space
- $[\psi(x), \psi^\dagger(y)] = \delta(x - y)$ acts on the physical QFT Hilbert space

Idea: A generalized coherent state

Computations

Some correlation functions

$$\left\langle \hat{\psi}(x)^\dagger \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{TL}(R \otimes \overline{R})\right]$$

$$\left\langle \hat{\psi}(x)^\dagger \hat{\psi}(0)^\dagger \hat{\psi}(0) \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{T(L-x)}(R \otimes \overline{R}) e^{Tx}(R \otimes \overline{R})\right]$$

$$\left\langle \hat{\psi}(x)^\dagger \left[-\frac{d^2}{dx^2}\right] \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{TL}([Q, R] \otimes [\overline{Q}, \overline{R}])\right]$$

with $T = Q \otimes \mathbb{1} + \mathbb{1} \otimes \bar{Q} + R \otimes \bar{R}$

Example

Lieb-Liniger Hamiltonian

$$\mathcal{H} = \int_{-\infty}^{+\infty} dx \left[\frac{d\hat{\psi}^\dagger}{dx}\frac{d\hat{\psi}}{dx} - \mu \hat{\psi}^\dagger \hat{\psi} + c \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi}\right]$$

Solve by **minimizing**: $\langle Q, R | \mathcal{H} | Q, R \rangle = f(Q, R)$

Standard CMPS and ϕ^4

Applying cMPS to the ϕ^4 Hamiltonian

$$\langle Q, R|\hat{h}_{\phi^4}|Q, R\rangle = \infty$$

Oh no!

The short distance behavior of cMPS is the wrong one, even the free theory is hard to approximate.

Going relativistic

Infusing some “high-energy” knowledge into tensor networks

Towards relativistic CMPS

Local basis in position of the QFT: $\psi^\dagger$, ϕ, π, $|0\rangle_\psi$
Diagonal basis of the free part: $a_k^\dagger$, $|0\rangle_a$

Bogoliubov transform

Go from $\hat{\psi}(x), \hat{\psi}^\dagger(x)$ to $a(p), a^\dagger(p)$ with

$$a(p) = \frac{1}{\sqrt{2}}\left(\sqrt{\omega_p}\,\hat{\phi}(p) + \frac{\hat{\pi}(p)}{\sqrt{\omega_p}}\right) \quad \text{with} \quad \omega_p = \sqrt{p^2 + m^2}$$

which yields

$$H_0 = \int \mathrm{d}p\, \omega_p\, \frac{1}{2}\left(a_p^\dagger a_p + a_p a_p^\dagger\right)$$

Go from $|0\rangle_\psi$ to $|0\rangle_a$
and
Go from $\psi(x)$ to $a(x) = \int \mathrm{d}p\; a(p)\mathrm{e}^{ipx} \neq \psi(x)$

Relativistic CMPS

Definition

$$|R, Q\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q \otimes \mathbb{1} + R \otimes a^{\dagger}(x)\right]\right\}|0\rangle_a$$

Some properties

1. $|0, 0\rangle = |0\rangle_a$ is the ground state of H_0 hence exact CFT UV fixed point (because interaction super-renormalizable)
2. $\langle Q, R|h_{\phi^4}|Q, R\rangle$ is finite for all Q, R (not trivial)

Consequence on the Hamiltonian

Hamiltonian density in $a(x)$ basis

H is local in $\psi(x)$, not in $a(x)$...

$$
\begin{aligned}
H = &\int \mathrm{d}x_1 \mathrm{d}x_2 D(x_1 - x_2) a^\dagger(x_1) a(x_2) \\
&+ \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 K(x_1, x_2, x_3, x_4) a(x_1) a(x_2) a(x_3) a(x_4) + 4a^\dagger aaa + 3a^\dagger a^\dagger aa \\
&+ \mathrm{h.c.}
\end{aligned}
$$

But fortunately exponentially decreasing: K is horrible, but decays $\propto e^{-m|x|}$.

The nightmarish optimization

Compute $e_0 = \langle Q, R | h_{\phi^4} | Q, R \rangle$ and $\nabla_{Q,R} e_0$

1. Contains an algebraic part identical to standard cMPS
2. Involves horrible quadruple integrals without analytic solutions

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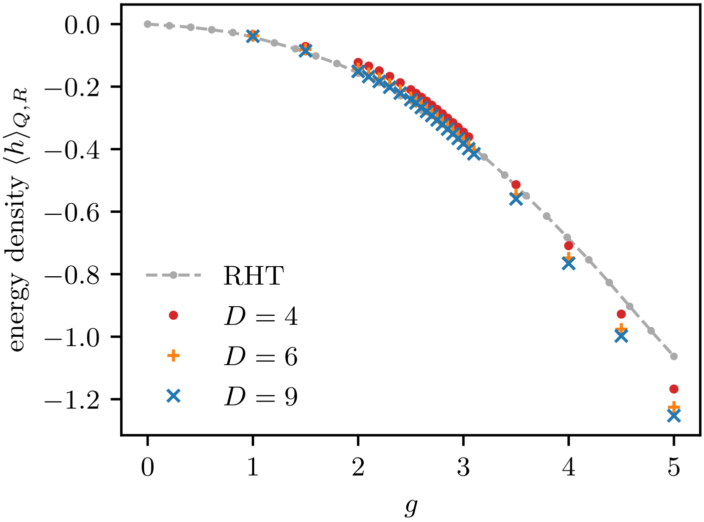
Optimization with naive gradient descent, BFGS, or conjugate gradient leads to plateaus $\implies$ does not work

One needs to do TDVP (i.e. variational optimization with a metric), slightly more complicated but "standard" and works. Equivalent with imaginary time evolution with large time-step.

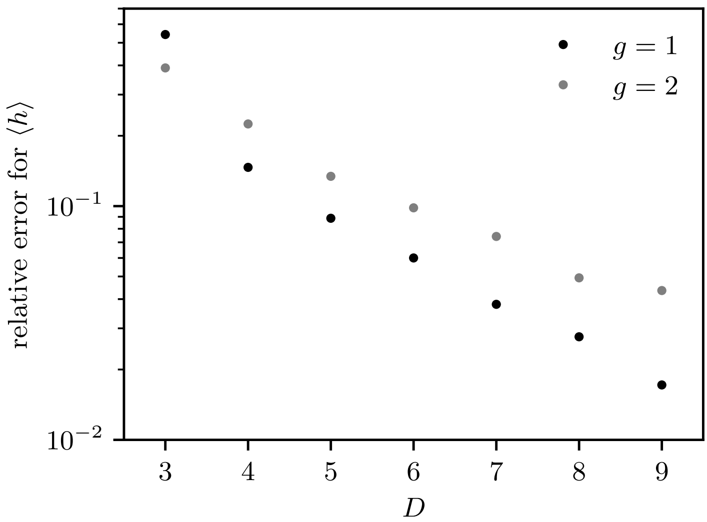
Results and discussion

Results



Compared with the Renormalized Hamiltonian Truncation results of Rychkov and Vitale from 2015.

Results



Compared with the "high precision" Renormalized Hamiltonian Truncation results of Elias Miro, Rychkov, and Vitale from 2017 for $g = 1$ and $g = 2$

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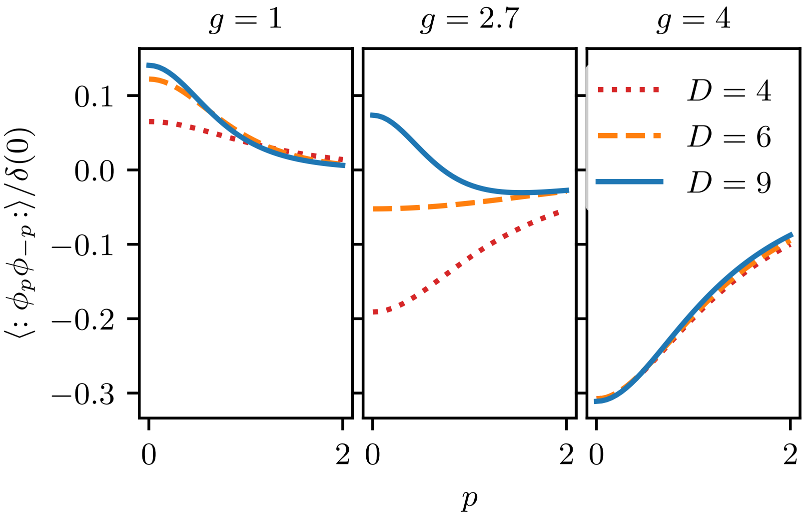

Normal ordered momentum two point function $\langle : \phi_p \phi_{-p} : \rangle_{Q,R}$

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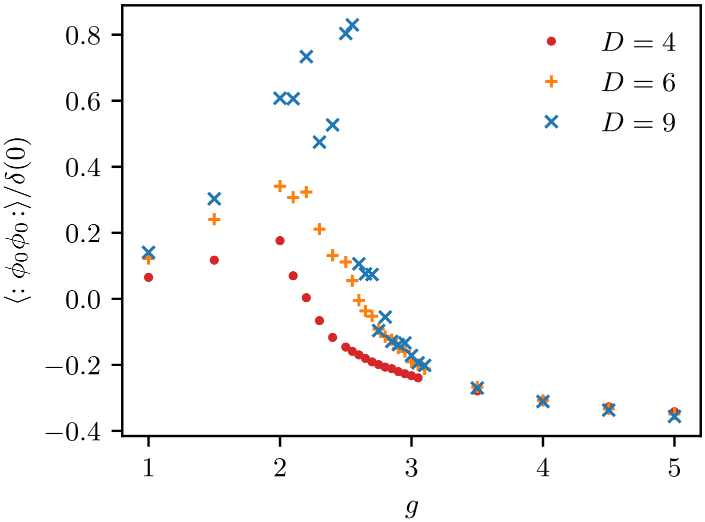

Normal ordered momentum two point function at zero momentum $\langle : \phi_0\phi_0 : \rangle_{Q,R}$

Comparison with renormalized Hamiltonian truncation

Ren. Hamiltonian truncation

IR cutoff L, energy truncation E_T

- Uses a vector space
- Function to minimize is quadratic, hence linear problem
- Number of parameters $\propto \mathrm{e}^{L \times E_T}$
- Error $\propto 1/E_T^3$
- Spectrum easy

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Relativistic CMPS

entanglement truncation D

- Uses a manifold
- Minimization is a priori non-trivial but doable
- Number of parameters $\propto D^2$
- Error $o(1/D^\alpha)$, $\forall\ \alpha$ (folklore)
- Fixed t correl. functions easy

Note: real world not asymptotic. RCMPS has expensive prefactors, and RHT can use reliable extrapolations

Extensions

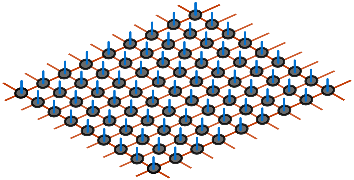
- To other bosonic theories in $1+1$ with poly $V(\phi)$ $\rightarrow$ easy
- To fermionic theories in $1+1$ $\rightarrow$ feasible
- To $2+1$ and $3+1$ dimensions $\rightarrow$ very difficult
  (lattice tensor networks will probably rule in $1+1$ and $2+1$ for numerics)

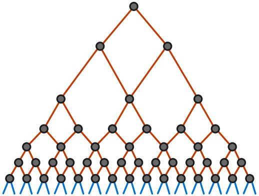
Summary

1. New ansatz for $1+1$ relativistic QFT
2. No cutoff, UV or IR (a first?)
3. UV is captured exactly even at $D=0$
4. Efficient (cost poly D, error superpoly $1/D$)
5. Rigorous (variational)

Bonus: more on tensor network states

Tensor network states: a tool





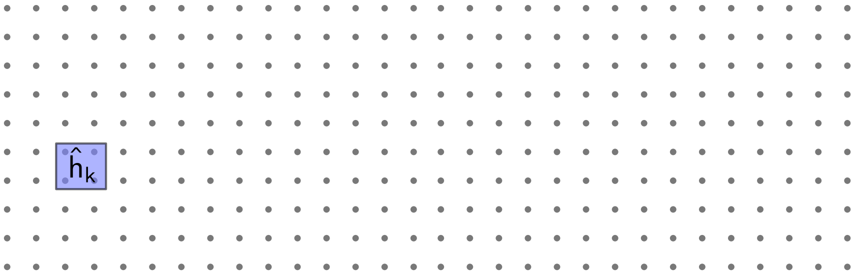
Applications

- Quantum information theory
- Statistical Mechanics
- Quantum gravity
- **Many-body quantum**

Negative theology

- **Not** covariant/geometric objects $g_{\mu\nu}$ or $R^{\sigma}_{\mu\nu\kappa}$
- **Not** tensor **models** [Rivasseau, Gurau, ...]

Many-body problem



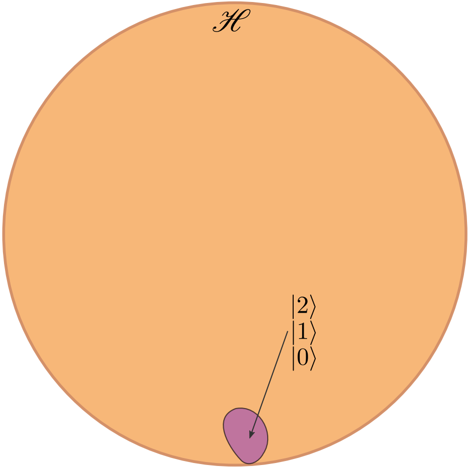
Problem

Finding low energy states of

$$\hat{H} = \sum_{k=1}^{N} \hat{h}_k$$

is **hard** because $\dim \mathscr{H} \propto D^N$

Variational optimization



Generic (spin $d/2$) state $\in \mathscr{H}$:

$$|\psi\rangle = \sum_{i_1, i_2, \cdots, i_n} c_{i_1, i_2, \cdots, i_N} |i_1, \cdots, i_N\rangle$$

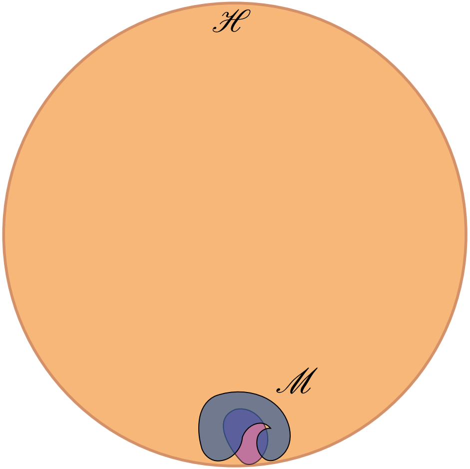
Exact variational optimization

To find the ground state:

$$|0\rangle = \min_{|\psi\rangle \in \mathscr{H}} \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

- $\dim \mathscr{H} = d^N$

Variational optimization



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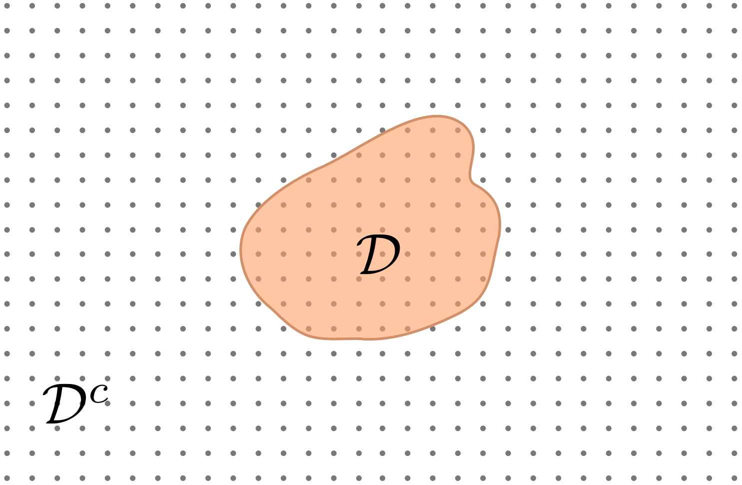
Approx. variational optimization

To find the ground state:

$$|0\rangle = \min_{|\psi\rangle \in \mathscr{M}} \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

- $\dim \mathscr{M} \propto \mathrm{Poly}(N)$ or fixed

Interesting states are weakly entangled



Low energy state

$|\psi\rangle = |0\rangle$ or $|1\rangle$ ...

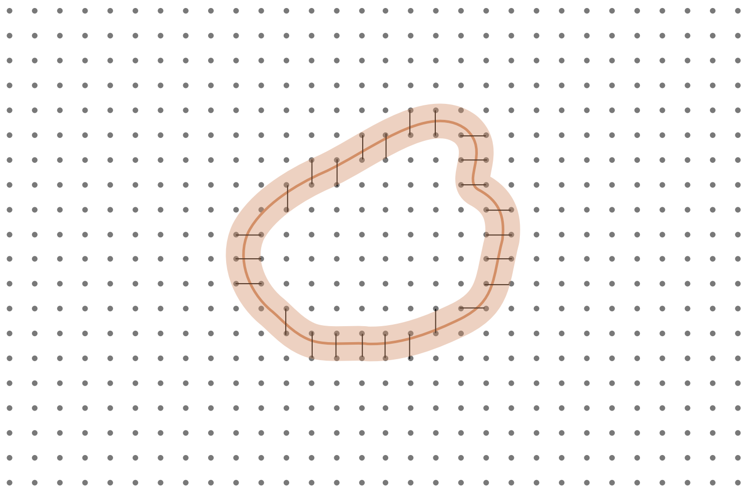
Reduced density matrix
$\rho = \mathrm{tr}_{\mathcal{D}^c}\left[|\psi\rangle\langle\psi|\right]$

Entanglement entropy
$S = -\mathrm{tr}\left[\rho \log \rho\right]$

Area law

$$S \propto |\partial \mathcal{D}|$$

Interesting states are weakly entangled



Low energy state

$|\psi\rangle = |0\rangle$ or $|1\rangle$ ...

Reduced density matrix

$\rho = \mathrm{tr}_{\mathcal{D}^c}\left[|\psi\rangle\langle\psi|\right]$

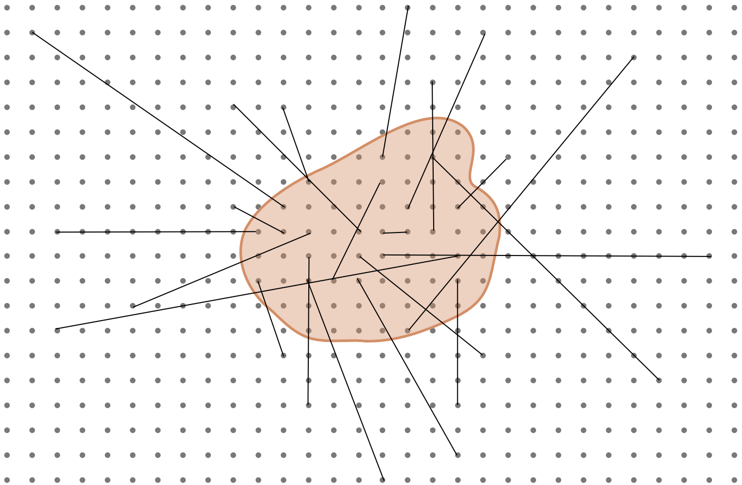
Entanglement entropy

$S = -\mathrm{tr}\left[\rho \log \rho\right]$

Area law

$$S \propto |\partial \mathcal{D}|$$

Typical states are strongly entangled



Random state

$|\psi\rangle = U_{\text{Haar}}|\text{trivial}\rangle$

Reduced density matrix

$\rho = \text{tr}_{\mathcal{D}^c}\left[|\psi\rangle\langle\psi|\right]$

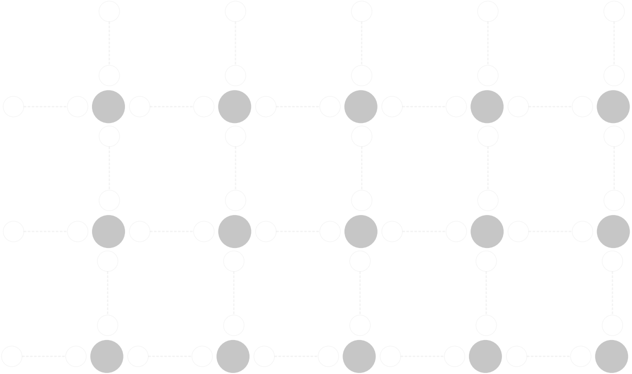
Entanglement entropy

$S = -\text{tr}\left[\rho \log \rho\right]$

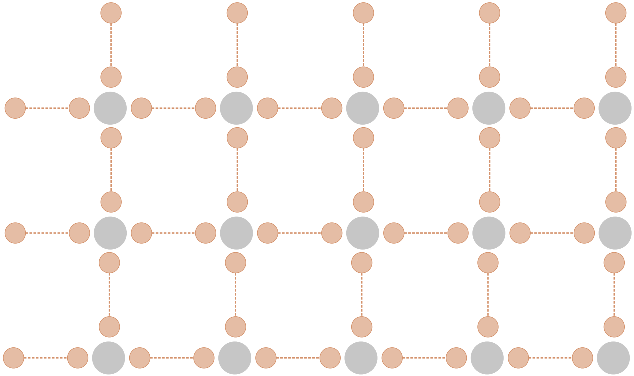
Volume law

$S \propto |\mathcal{D}|$

Constructing weakly entangled states



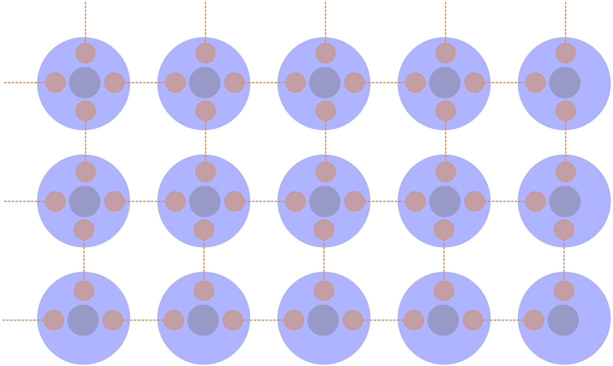
Constructing weakly entangled states



1. Put auxiliary **maximally entangled** states between sites

 $= \sum_{j=1}^{D} |j\rangle|j\rangle$

Constructing weakly entangled states



1. Put auxiliary **maximally entangled** states between sites

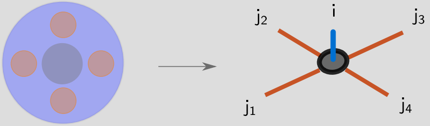
$$= \sum_{j=1}^{D} |j\rangle|j\rangle$$

2. Map to initial Hilbert space on each site

$$= A : \mathbb{C}^{4D} \to \mathbb{C}^{d}$$

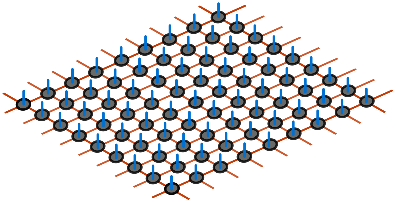
Tensor network states: definition

Why “tensor” network?



$$A : \mathbb{C}^{4D} \rightarrow \mathbb{C}^{d} \qquad \longrightarrow \qquad A^{i}_{j_1, j_2, j_3, j_4}$$

$|A\rangle =$



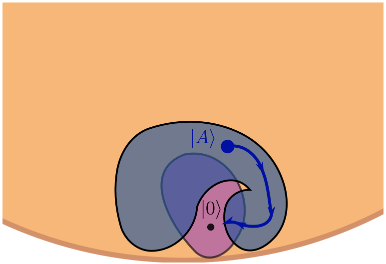
with tensor contractions on links

Optimization

Find best A for fixed χ $\quad (d \times D^4$ coeff.)

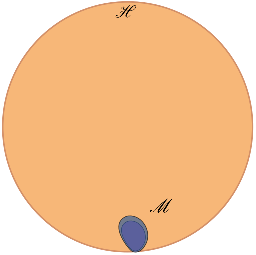
$$E_0 \simeq \min_{A} \frac{\langle A|\widehat{H}|A\rangle}{\langle A|A\rangle}$$

for example go down $\frac{\partial E}{\partial A^{i}_{j_1, j_2, j_3, j_4}}$

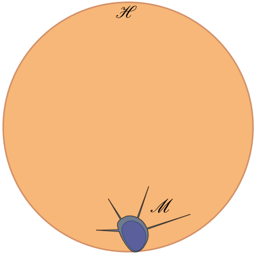


Some facts

1 spatial dimension



$\geqslant 2$ spatial dimension



Theorems (colloquially)

1. For gapped H, tensor network states $|A\rangle$ approximate well $|0\rangle$ as D increases
2. **All** $|A\rangle$ are ground states of local gapped H

Folklore

1. For gapped H, tensor network states $|A\rangle$ approximate well $|0\rangle$ as D increases
2. **Most** $|A\rangle$ are ground states of local gapped H