Relativistic continuous matrix product states

1+1d RQFT without cutoff

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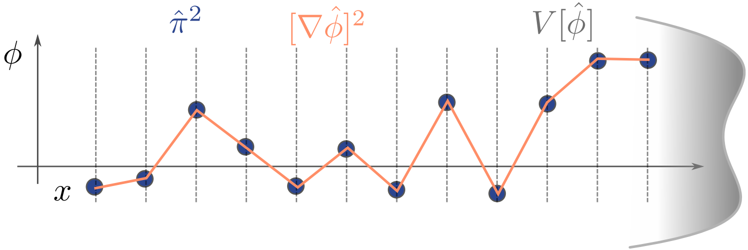

University of Ghent
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ϕ^4_2 - pile of dirt

QCD - Everest

$\mathcal{N} = 4$ *SYM* - Chrysler building

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Goal - ideal - philosophy: an apology of the pile of dirt approach

Abandon analytical solutions, but find robust methods that can solve simple QFTs non-perturbatively and, if possible, to machine precision, *without cheating*.

more on this on tilloy.wordpress.com

Fundamental Physics with tensor networks

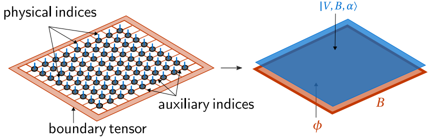
To apply tensor networks to "fundamental" theories, we need to understand:

1. Weird degrees of freedom (Gauge theories)
2. The continuum limit
3. Peculiarities of relativistic Hamiltonians (CFT at short distance)

What we did so far on the continuum at MPQ

"Analytical" Continuous tensor networks

1. Introduce a "good" definition of continuous tensor network in $d \geqslant 2$ (with Ignacio)



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$\longrightarrow$ both non-relativistic, "condensed-matter QFT"

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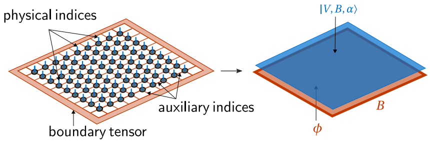

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"Numerical" Continuous tensor networks

1. Discretize ϕ_2^4 on a super-fine lattice, solve with standard methods, extrapolate the result to the continuum limit (with Clément)

True vs Effective QFT

Against the "why bother since there is always a cutoff?"

Effective QFT

The theory has a momentum/energy cutoff Λ large but finite $\Lambda \gg m$, where m is the gap.

The fundamental theory is not known, but in perturbation theory, one can take $\Lambda \to \infty$ term by term to get a good approximation of physics at scale m.

Examples

1. QED with matter
2. ϕ_4^4

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True QFT

The limit $\Lambda \to +\infty$ can be taken exactly, and the theory is valid "all the way down".

All quantities exist non-perturbatively in the limiting theory, for arbitrarily high energy. No cutoff whatsoever in principle.

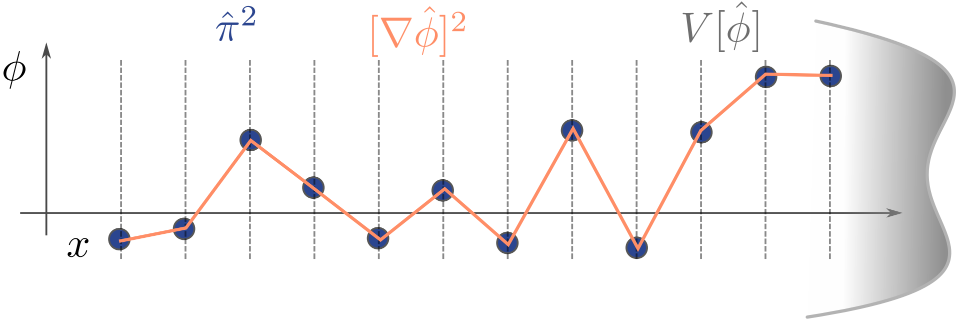
Examples

1. QCD without too much matter
2. ϕ^4_2 and ϕ^4_3
3. Sine-Gordon, Gross-Neveu, etc.

Outline

1. ϕ^4 theory – the condensed matter way
2. Divergences and standard resolution
3. ϕ^4 theory – the rigorous way
4. Illustration on lattice based approach
5. cMPS to the rescue?
6. relativistic cMPS and preliminary results

Intuitive definition: canonical quantization



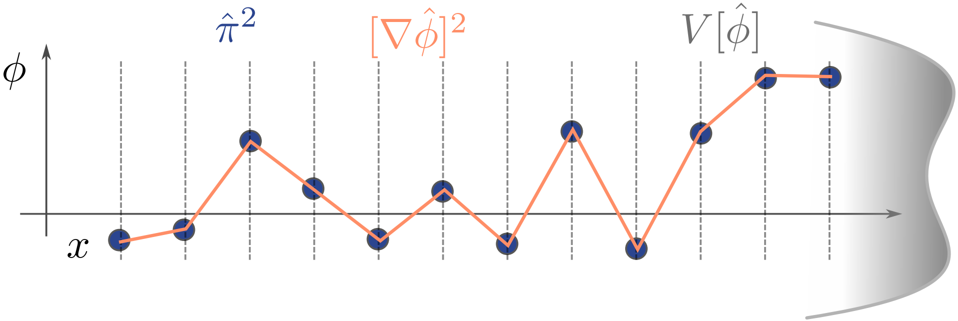
Hamiltonian

A continuum of nearest neighbor coupled anharmonic oscillators

$$\hat{H} = \int_{\mathbb{R}^d} \mathrm{d}^d x \quad \underbrace{\frac{\hat{\pi}(x)^2}{2}}_{\text{on-site inertia}} + \underbrace{\frac{[\nabla\hat{\phi}(x)]^2}{2}}_{\text{spatial stiffness}} + \underbrace{V(\hat{\phi}(x))}_{\text{on-site potential}}$$

with canonical commutation relations $[\hat{\phi}(x), \hat{\pi}(y)] = i\delta^d(x-y)\mathbb{1}$ (*i.e.* bosons)

Intuitive definition



Hilbert space

Fock space $\mathscr{H}_{\mathrm{QFT}} = \mathscr{F}[L^2(\mathbb{R}^d)]$ – just like $x, p \to (a, a^\dagger)$ do $\hat{\pi}, \hat{\phi} \to \hat{\psi}, \hat{\psi}^\dagger$

$$|\Psi\rangle = \sum_{n=0}^{+\infty} \int \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \underbrace{\varphi_n(x_1, x_2, \cdots, x_n)}_{\text{wave function}} \underbrace{\hat{\psi}^\dagger(x_1)\hat{\psi}^\dagger(x_2)\cdots\hat{\psi}^\dagger(x_n)}_{\text{local oscillator creation}} |\mathrm{vac}\rangle$$

What are the problems - Hilbert space approach

The Hamiltonian is ill defined on all states in the Hilbert space because of infinite zero point energy *i.e.* terms $\propto \widehat{\psi}(x)\widehat{\psi}^\dagger(x)$

$$\langle\Psi_1|\widehat{H}|\Psi_2\rangle = \pm\infty \quad \text{and even} \quad \langle\text{vac}|\widehat{H}|\text{vac}\rangle \propto \delta^d(0) = +\infty$$

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If the divergent vacuum terms are removed, the Hamiltonian is not bounded from below

$$\forall\ |\Psi\rangle \in \mathscr{H},\ \langle \Psi | \hat{H}_{\text{finite}} | \Psi \rangle = \text{finite but} \quad \exists\ \Psi_n \text{ s.t. } \lim_{n \to +\infty} \langle \Psi_n | H_{\text{finite}} | \Psi_n \rangle = -\infty$$

and worse

$$|0\rangle := \lim_{n \to +\infty} |\Psi_n\rangle \notin \mathscr{H}$$

How are they are solved in the free case - Hamiltonian

Bogoliubov transform

Go from $\hat{\psi}(x), \hat{\psi}^\dagger(x)$ to $a(p), a^\dagger(p)$ with

$$a(p) = \frac{1}{\sqrt{2}}\left(\sqrt{\omega_p}\,\hat{\phi}(p) + \frac{\hat{\pi}(p)}{\sqrt{\omega_p}}\right) \quad \text{with} \quad \omega_p = \sqrt{p^2 + m^2}$$

which yields

$$H_0 = \int \mathrm{d}p\,\omega_p\,\frac{1}{2}\left(a_p^\dagger a_p + a_p a_p^\dagger\right)$$

Solution

- Take $H_{\mathrm{QFT}} \equiv\, : H :_a$
- $|\text{ground state}\rangle = |\text{vacuum}\rangle_a$
- $\mathscr{H}$ built from $a_{p_1}^\dagger \cdots a_{p_n}^\dagger |\text{vacuum}\rangle_a$

This solves the problematic free part exactly, and allows to define a finite interaction

Rigorous operator definition of ϕ_2^4

Renormalized ϕ_2^4 theory:

$$H = \int dx \, \frac{:\pi^2:_a}{2} + \frac{:(\nabla\phi)^2:_a}{2} + \frac{m^2}{2} :\phi^2:_a + g :\phi^4:_a$$

note that $:\diamondsuit:_a$ depends on m!

1. Rigorously defined relativistic QFT without cutoff (Wightman QFT)
2. Vacuum energy density finite
3. Very difficult to solve unless $g \ll m^2$ (perturbation theory)
4. Phase transition around $f_c = \frac{g}{4m^2} = 11$ i.e. $g \simeq 2.7$ in mass units

Ways to solve ϕ_2^4

With a lattice of size a (UV cutoff) and fixed number of sites N (IR cutoff)

- Monte-Carlo
- Tensor network renormalization

With a lattice of size a (UV cutoff) and no IR cutoff

- Uniform MPS

With continuous space, an energy cutoff Λ (UV) and an IR cutoff

- Hamiltonian truncation

Without cutoff

- Perturbation theory + Borel-Padé resummation

Lattice ϕ_2^4

Discretize the action:

$$S(\phi) = \sum_{\langle i,j \rangle} \underbrace{\frac{(\phi_i - \phi_j)^2}{2a^2}}_{(\nabla\phi)^2/2} a^2 + \sum_i \frac{1}{2}\mu_a^2 \phi_i^2 + \frac{1}{4}\lambda_a \phi_i^4$$

Taking the limit

The right way to get the continuum limit is to take:

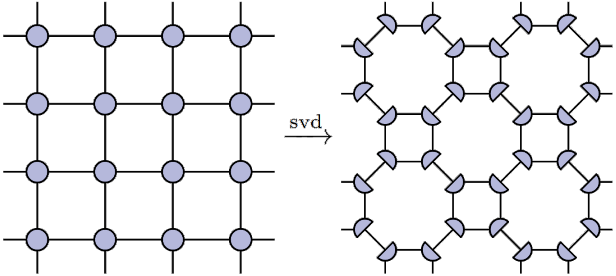
$$\mu_a = \mu a^2 + \frac{3}{2}\log(a) a^2 \lambda$$
$$\lambda_a = \lambda a^2$$

which is equivalent to normal ordering

Basically, at first order in perturbation theory, the ϕ^4 term behaves like a ϕ^2 term times a log divergent constant.

Example with tensor network renormalization

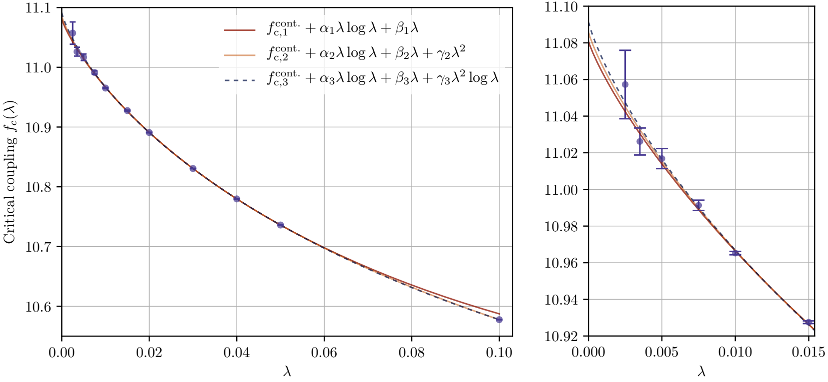
Done with Clément [late 2019 – early 2020]
Discretize ϕ, write $Z = \sum S(\phi)$ as a tensor network and contract it with TRG + GILT



Technically: UV cutoff (lattice) and IR cutoff (number of RG steps)

Example with tensor network renormalization

Continuum limit taken **numerically**



More costly as the UV cutoff gets small because:

1. Field becomes unbounded at short distance $\rightarrow$ large starting bond dimension
2. More RG steps (with max χ) to get to the same scale

Limitation of numerical continuum limit

The "numerical" continuum limit is expensive for relativistic QFT. Is it a problem of local basis choice?

No:

1. UV fixed point is a free CFT, so technically continuum of singular values
2. Interaction is super renormalizable / strongly relevant, so its impact on the tensors $\to 0$ in continuum limit

$\Longrightarrow$ even theory independent: would apply to QCD (asymptotic freedom), but worse for super-renormalizable theories

Continuous Matrix Product States

Type of ansatz for bosons on a fine grained $d = 1$ lattice

- Matrices $A_{i_k}(x)$ where the index i_k corresponds to $\psi^{\dagger i_k}(x)|0\rangle$ in physical space.

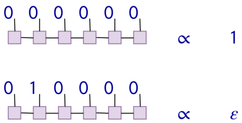
Informal cMPS definition

$$A_0 = \mathbb{1} + \varepsilon Q$$
$$A_1 = \varepsilon R$$
$$A_2 = \frac{(\varepsilon R)^2}{\sqrt{2}}$$
$$\ldots$$
$$A_n = \frac{(\varepsilon R)^n}{\sqrt{n}}$$

so we go from ∞ to 2 matrices

Fixed by:

- Finite particle number



- Consistency

Continuous Matrix Product States

Definition

$$|Q, R, \omega\rangle = \langle\omega_L|\mathcal{P}\exp\left\{\int_0^L dx\ Q\otimes\mathbb{1} + R\otimes\psi^\dagger(x)\right\}|\omega_R\rangle\ |0\rangle_\psi$$

- Q, R are $D\times D$ matrices,
- $|\omega_L\rangle$ and $|\omega_R\rangle$ are boundary vectors $\in\mathbb{C}^D$, for p.b.c. $\langle\omega_L|\cdot|\omega_R\rangle\to\mathrm{tr}[\,\cdot\,]$
- $[\psi(x), \psi^\dagger(y)] = \delta(x-y)$

Idea:

$$\begin{aligned} A(x) &\simeq A_0\mathbb{1} + A_1\psi^\dagger(x) \\ &\simeq \mathbb{1}\otimes\mathbb{1} + \varepsilon Q\otimes\mathbb{1} + \varepsilon R\otimes\psi^\dagger(x) \\ &\simeq \exp\left[\varepsilon\left(Q\otimes\mathbb{1} + R\otimes\psi^\dagger(x)\right)\right] \end{aligned}$$

Computations

Some correlation functions

$$\left\langle \hat{\psi}(x)^\dagger \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{TL} (R \otimes \overline{R}) \right]$$

$$\left\langle \hat{\psi}(x)^\dagger \hat{\psi}(0)^\dagger \hat{\psi}(0) \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{T(L-x)} (R \otimes \overline{R}) e^{Tx} (R \otimes \overline{R}) \right]$$

$$\left\langle \hat{\psi}(x)^\dagger \left[-\frac{d^2}{dx^2} \right] \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{TL} ([Q, R] \otimes [\overline{Q}, \overline{R}]) \right]$$

with $T = Q \otimes \mathbb{1} + \mathbb{1} \otimes \bar{Q} + R \otimes \bar{R}$

Example

Lieb-Liniger Hamiltonian

$$\mathcal{H} = \int_{-\infty}^{+\infty} dx \left[\frac{d\hat{\psi}^\dagger}{dx} \frac{d\hat{\psi}}{dx} - \mu \hat{\psi}^\dagger \hat{\psi} + c \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \right]$$

Solve by **minimizing**: $\langle Q, R | \mathcal{H} | Q, R \rangle = f(Q, R)$

Standard CMPS and ϕ^4

Applying cMPS to the ϕ^4 Hamiltonian

$$\langle Q, R|\widehat{h}_{\phi^4}|Q, R\rangle = \infty$$

Oh no!

The short distance behavior of cMPS is the wrong one, even the free theory is hard to approximate.

Feynman's objection

Feynman's requirement for variational wavefunctions in RQFT

1. Extensive
2. Computable expectation values
3. Not oversensitive to the UV

CMPS do 1 and 2 but struggle with 3.

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Haegeman-Cirac-Osborne-Verschelde-Verstraete fix of 2010: regulate the UV by adding a Lagrange multiplier in the Hamiltonian

$$H \to H + \frac{1}{\Lambda^2}\text{regulator}$$

Towards relativistic CMPS

Local basis in position of the QFT: $\psi^\dagger$, ϕ, π, $|0\rangle_\psi$
Diagonal basis of the free part: $a_k^\dagger$, $|0\rangle_a$

Bogoliubov transform

Go from $\hat{\psi}(x), \hat{\psi}^\dagger(x)$ to $a(p), a^\dagger(p)$ with

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Go from $|0\rangle_\psi$ to $|0\rangle_a$
and
Go from $\psi(x)$ to $a(x) = \int \mathrm{d}p\; a(p) \mathrm{e}^{ipx} \neq \psi(x)$

Relativistic CMPS

Definition

$$|R, Q\rangle = \operatorname{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q\otimes \mathbb{1} + R\otimes a^\dagger(x)\right]\right\}|0\rangle_a$$

Some properties

1. $|0,0\rangle = |0\rangle_a$ is the ground state of H_0 hence exact CFT UV fixed point (because interaction super-renormalizable)
2. $\langle Q, R|h_{\phi^4}|Q, R\rangle$ is finite for all Q, R (not trivial)

Consequence on the Hamiltonian

Hamiltonian density in $a(x)$ basis

H is local in $\psi(x)$, not in $a(x)$...

$$
\begin{aligned}
H = &\int \mathrm{d}x_1 \mathrm{d}x_2 D(x_1 - x_2) a^\dagger(x_1) a(x_2) \\
&+ \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 K(x_1, x_2, x_3, x_4) a(x_1) a(x_2) a(x_3) a(x_4) + 4a^\dagger aaa + 3a^\dagger a^\dagger aa \\
&+ \text{h.c.}
\end{aligned}
$$

But fortunately exponentially decreasing: K is horrible, but decays $\propto e^{-m|x|}$.

The nightmarish optimization

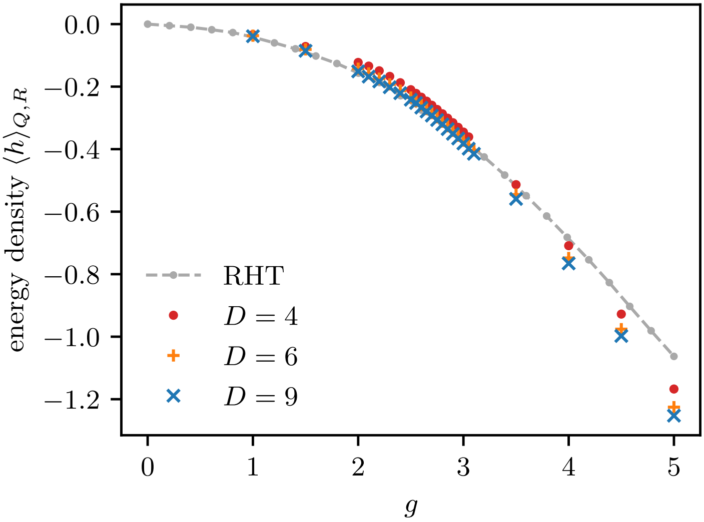
Compute $e_0 = \langle Q, R | h_{\phi^4} | Q, R \rangle$ and $\nabla_{Q,R} e_0$

1. Contains an algebraic part identical to standard cMPS
2. Involves horrible quadruple integrals without analytic solutions

Optimization with naive gradient descent, BFGS, or conjugate gradient leads to plateaus $\implies$ does not work

One needs to do TDVP (i.e. variational optimization with a metric), slightly more complicated but "standard" (in Ghent at least) and works. Equivalent with imaginary time evolution with large time-step.

Results



Compared with the Renormalized Hamiltonian Truncation results of Rychkov and Vitale from 2015.

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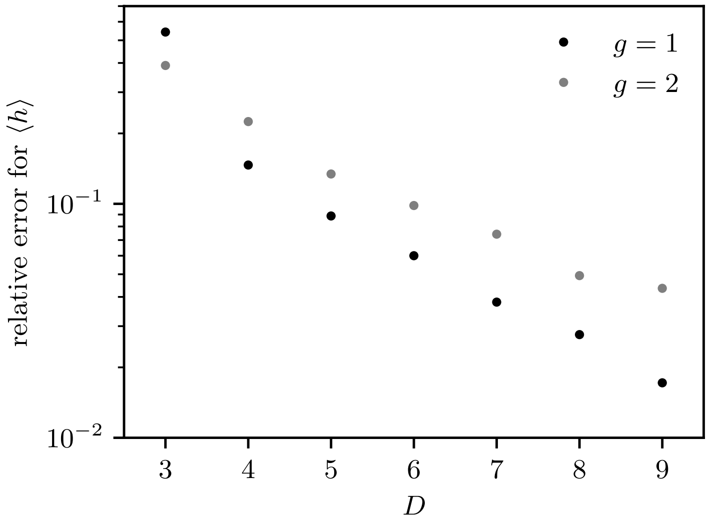

Compared with the “high precision” Renormalized Hamiltonian Truncation results of Elias Miro, Rychkov, and Vitale from 2017 for $g = 1$ and $g = 2$

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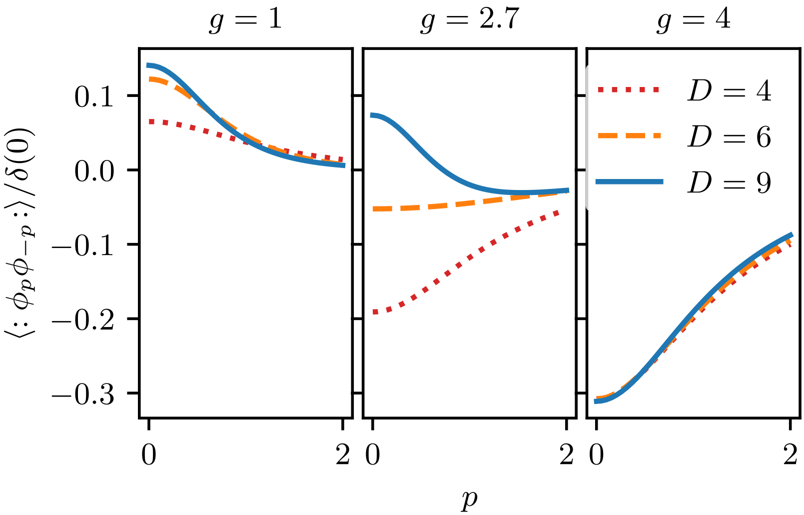

Normal ordered momentum two point function $\langle : \phi_p \phi_{-p} : \rangle_{Q,R}$

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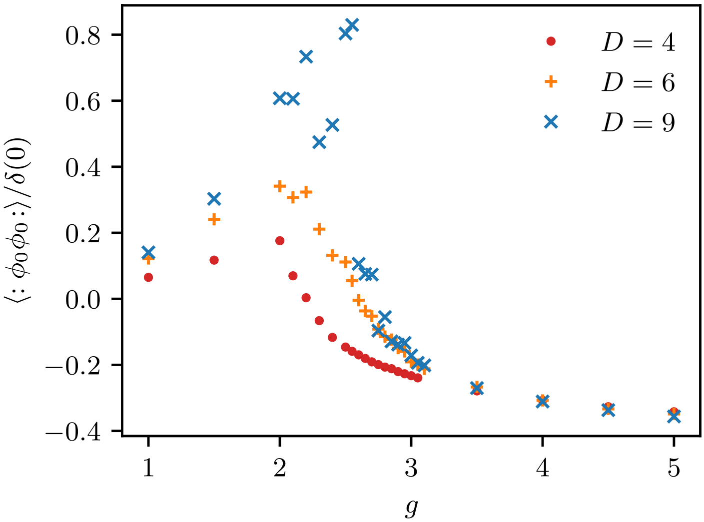

Normal ordered momentum two point function at zero momentum $\langle : \phi_0 \phi_0 : \rangle_{Q,R}$

What now

On the immediate numerical side:

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2. Do fermions: are there new regularity conditions?
3. CMERA for relativistic QFT? (i.e. a flow between UV and IR CFT)

Summary

1. New ansatz for $1 + 1$ relativistic QFT
2. No cutoff, UV or IR (a first?)
3. UV is captured exactly even at $D = 0$
4. Rigorous (variational)