Variational method in relativistic QFT

without cutoff

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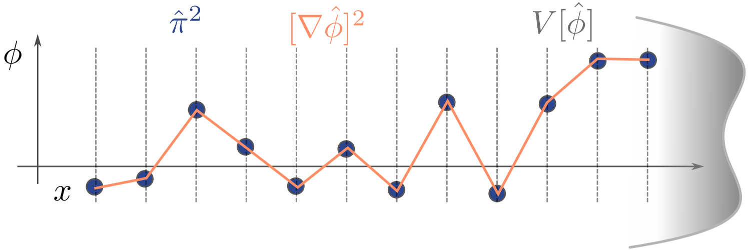

Thursday seminar of QUANTIC, Paris
March 11th, 2021

The problem of quantum field theory

Basics:

- Quantum field theory $=$ most fundamental description of Nature
- Continuously infinite many-body problem

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- Free QFT are easy (both to define and solve)
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Bruteforce way:

- Non-perturbative computations are doable with lattice Monte-Carlo
- But many quantities of interest out of reach even with exascale computing in lattice QCD

Quantum field theory: a bit of philosophy

Two ways to attack *real world* quantum field theories non-perturbatively

1. Start **simpler** so that it becomes **simpler** [*e.g.* ϕ_2^4]
2. Start **more complex** so that it becomes **simpler** [*e.g.* $\mathcal{N} = 4$ *SYM*]

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ϕ_2^4 - pile of dirt

QCD - Everest

$\mathcal{N} = 4$ *SYM* - Chrysler building

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Goal - ideal - philosophy: an apology of the pile of dirt approach

Abandon analytical solutions, but find robust methods that can solve simple QFTs non-perturbatively and, if possible, to machine precision, *without cheating*.

more on this on tilloy.wordpress.com

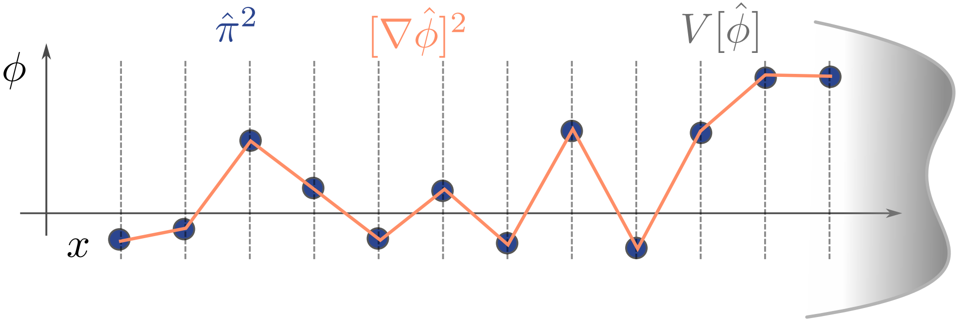
Outline

1. ϕ_2^4 for beginners
2. The variational method
3. Matrix product states and their continuum limit
4. Going relativistic
5. Results and discussion

ϕ_2^4 for beginners

and condensed matter theorists

Intuitive definition: canonical quantization



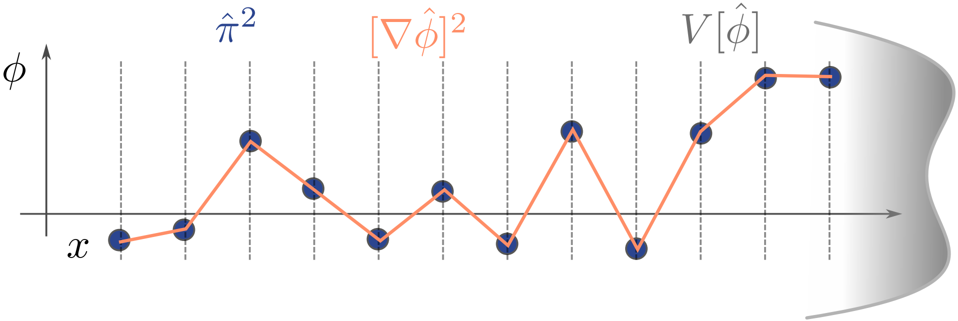
Hamiltonian

A continuum of nearest neighbor coupled anharmonic oscillators

$$\hat{H} = \int_{\mathbb{R}^d} \mathrm{d}^d x \quad \underbrace{\frac{\hat{\pi}(x)^2}{2}}_{\text{on-site inertia}} + \underbrace{\frac{[\nabla\hat{\phi}(x)]^2}{2}}_{\text{spatial stiffness}} + \underbrace{V(\hat{\phi}(x))}_{\text{on-site potential}}$$

with canonical commutation relations $[\hat{\phi}(x), \hat{\pi}(y)] = i\delta^d(x-y)\mathbb{1}$ (*i.e.* bosons)

Intuitive definition



Hilbert space

Fock space $\mathscr{H}_{\text{QFT}} = \mathscr{F}[L^2(\mathbb{R}^d)]$ – just like $x, p \to (a, a^\dagger)$ do $\hat{\pi}, \hat{\phi} \to \hat{\psi}, \hat{\psi}^\dagger$

$$|\Psi\rangle = \sum_{n=0}^{+\infty} \int \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \underbrace{\varphi_n(x_1, x_2, \cdots, x_n)}_{\text{wave function}} \underbrace{\hat{\psi}^\dagger(x_1)\hat{\psi}^\dagger(x_2)\cdots\hat{\psi}^\dagger(x_n)}_{\text{local oscillator creation}} |\text{vac}\rangle$$

Why relativistic? $\rightarrow$ functional integral

Insert $\mathbb{1} = \int \mathcal{D}\phi \, |\phi\rangle\langle\phi|$ in expression for correlation functions and $t = i\tau$ gives

Functional integral representation

Representation of correlation functions in terms of random fields

$$\langle 0|\hat{\phi}(\tau_1, x_1) \cdots \hat{\phi}(\tau_n, x_n)|0\rangle := \int \phi(\tau_1, x_1) \cdots \phi(\tau_1, x_n) \; e^{-S(\phi)} \underbrace{\mathcal{D}\phi}_{\text{“Lebesgue measure”}}$$

with the action / weight where $\hat{\pi} \rightarrow \frac{d\phi}{d\tau}$

$$S(\phi) = \int d^d x \, d\tau \quad \underbrace{\frac{1}{2}\left[\frac{d\phi}{d\tau}\right]^2}_{\text{inertia a.k.a time stiffness}} + \underbrace{\frac{[\nabla\phi]^2}{2}}_{\text{spatial stiffness}} + \underbrace{V(\phi)}_{\text{on-site potential}}$$

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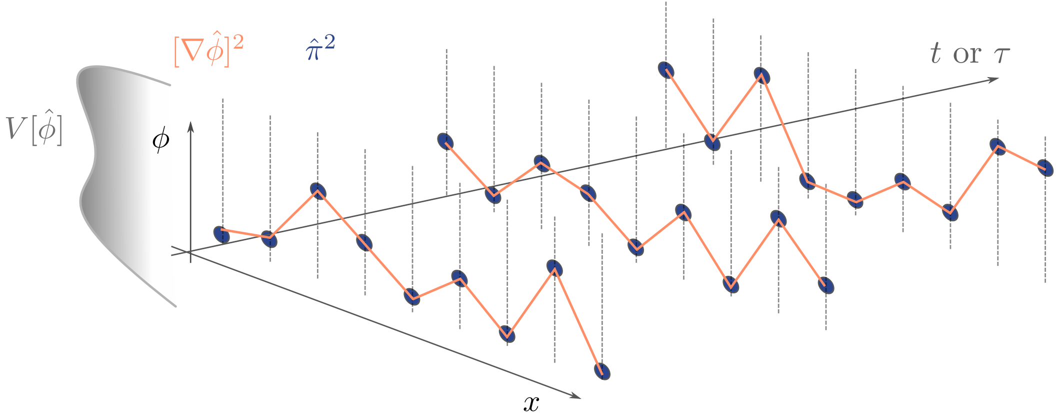
Inertia = time stiffness $\implies$ Euclidean rotation invariance $\implies$ Lorentz

Why relativistic? $\rightarrow$ functional integral



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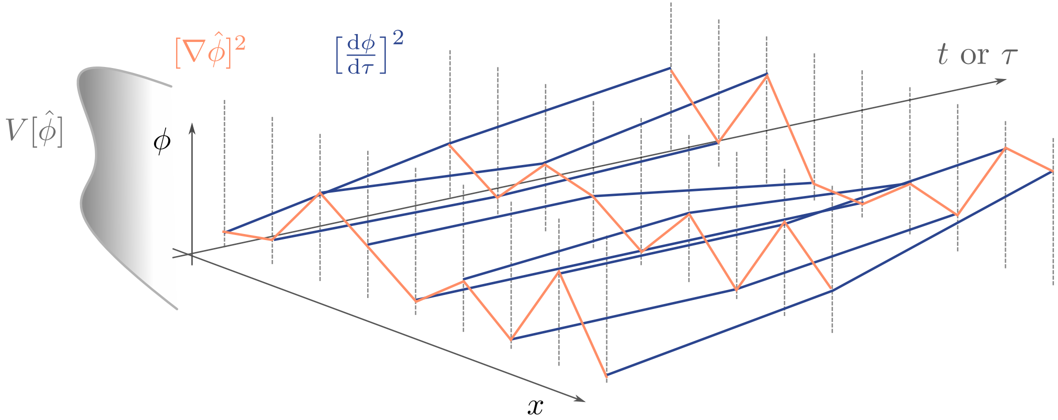

What are the problems - Hilbert space approach

The Hamiltonian is ill defined on all states in the Hilbert space because of infinite zero point energy *i.e.* terms $\propto \widehat{\psi}(x)\widehat{\psi}^\dagger(x)$

$$\langle \Psi_1|\widehat{H}|\Psi_2\rangle = \pm\infty \quad \text{and even} \quad \langle \text{vac}|\widehat{H}|\text{vac}\rangle \propto \delta^d(0) = +\infty$$

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If the divergent vacuum terms are removed, the Hamiltonian is not bounded from below

$$\forall\ |\Psi\rangle \in \mathscr{H},\ \langle\Psi|\widehat{H}_{\mathrm{finite}}|\Psi\rangle = \text{finite but} \quad \exists\ \Psi_n \text{ s.t. } \lim_{n\to+\infty}\langle\Psi_n|H_{\mathrm{finite}}|\Psi_n\rangle = -\infty$$

True vs Effective QFT

Against the "why bother since there is always a cutoff?"

Effective QFT

The theory has a momentum/energy cutoff Λ large but finite $\Lambda \gg m$, where m is the gap.

The fundamental theory is not known, but in perturbation theory, one can take $\Lambda \to \infty$ term by term to get a good approximation of physics at scale m.

Examples

1. QED with matter
2. ϕ_4^4

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True QFT

The limit $\Lambda \to +\infty$ can be taken exactly, and the theory is valid "all the way down".

All quantities exist non-perturbatively in the limiting theory, for arbitrarily high energy. No cutoff whatsoever in principle.

Examples

1. QCD without too much matter
2. ϕ_2^4 and ϕ_3^4
3. Sine-Gordon, Gross-Neveu, etc.

How problems are solved in the free case

Bogoliubov transform

Go from $\hat{\psi}(x), \hat{\psi}^\dagger(x)$ to $a(p), a^\dagger(p)$ with

$$a(p) = \frac{1}{\sqrt{2}}\left(\sqrt{\omega_p}\,\hat{\phi}(p) + \frac{\hat{\pi}(p)}{\sqrt{\omega_p}}\right) \quad \text{with} \quad \omega_p = \sqrt{p^2 + m^2}$$

which yields

$$H_0 = \int \mathrm{d}p\, \omega_p\, \frac{1}{2}\left(a_p^\dagger a_p + a_p a_p^\dagger\right)$$

Solution

- Take $H_{\text{QFT}} \equiv \, : H :_a$
- $|\text{free ground state}\rangle = |\text{vacuum}\rangle_a$
- $\mathscr{H}$ built from $a_{p_1}^\dagger \cdots a_{p_n}^\dagger |\text{vacuum}\rangle_a$

This solves the problematic free part exactly, and allows to define a finite interaction (in $1+1$)

Rigorous operator definition of ϕ^4_2

Renormalized ϕ^4_2 theory

$$H = \int \mathrm{d}x \, \frac{:\pi^2:_a}{2} + \frac{:(\nabla\phi)^2:_a}{2} + \frac{m^2}{2} :\phi^2:_a + g :\phi^4:_a$$

(note that $:\diamondsuit:_a$ depends on m)

1. Rigorously defined relativistic QFT without cutoff (Wightman QFT)
2. Vacuum energy density finite
3. Very difficult to solve unless $g \ll m^2$ (perturbation theory)
4. Phase transition around $f_c = \frac{g}{4m^2} = 11$ i.e. $g \simeq 2.7$ in mass units

"Skyscrapering" the pile of dirt

Scattering is complicated in ϕ_2^4, particle number is not conserved
$\rightarrow$ tweak the potential V to cancel all contributions terms

“Skyscrapering” the pile of dirt

Scattering is complicated in ϕ^4_2, particle number is not conserved
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Sinh-Gordon theory

$$H = \int \mathrm{d}x \, \frac{:\pi^2:_a}{2} + \frac{:(\nabla\phi)^2:_a}{2} + \frac{m^4}{g} :\cosh\left(\frac{g}{m^2}\phi\right):_a$$

- Gives ϕ^4 theory $+$ corrections by Taylor expanding cosh
- Infinitely many Feynman diagram
- Exactly solvable with Bethe Ansatz!
- But very peculiar / non-generic physics

The variational method

Solving the non-exactly solvable by guessing well

Ways to solve the non-exactly-solvable

The two main games in town

1. Perturbative expansions (+ Borel-Padé resummation)
2. Lattice Monte Carlo

Two “new” deterministic non-perturbative options:

1. Variational method $\rightarrow$ focus of today
2. Non-perturbative renormalization group (Kadanoff, FRG, Tensor RG, etc.)

The two new methods now rule on (low dimensional) lattice problems thanks to tensor networks $\rightarrow$ QFT?

The variational method

In the Hamiltonian formulation:

- Guess a **finite dimensional submanifold** $\mathscr{M}$ of the QFT Hilbert space $\mathscr{H}$
- Find the ground state by minimizing $\langle H \rangle$:

$$|\text{ground}\rangle \simeq |\psi\rangle = \underset{\mathscr{M}}{\text{argmin}} \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}$$

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Example: naive Hamiltonian truncation

With an IR cutoff, momenta are discrete. Take as submanifold $\mathscr{M}$ the **vector space** spanned by:

$$a_{k_1}^\dagger a_{k_2}^\dagger \cdots a_{k_r}^\dagger |0\rangle_a$$

where $r \leqslant r_{\text{max}}$ and $k \leqslant k_{\text{max}}$ (one possible truncation)

Feynman's objection

Feynman's requirement for variational wavefunctions in RQFT (1987)

1. **Extensive parameterization**
   Number of parameters $\propto L^\alpha$ at most for system size L
2. **Computable expectation values**
   ψ known $\implies \langle \mathcal{O}(x)\mathcal{O}(y)\rangle_\psi$ computable
3. **Not oversensitive to the UV**
   no runaway minimization where higher and higher momenta get fitted

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All methods so far break one at least:

- Hamiltonian truncation fails at 1 (but works fairly well through its renormalized refinements)
- Tensor networks succeed at 1 and 2 but fail (a priori) at 3

Haegeman-Cirac-Osborne-Verschelde-Verstraete fix of 2010: regulate the UV by adding a Lagrange multiplier in the Hamiltonian $H \to H + \frac{1}{\Lambda^2}\text{regulator}$

(Continuous) matrix product states

Taking the simplest tensor network and scaling it up to QFT

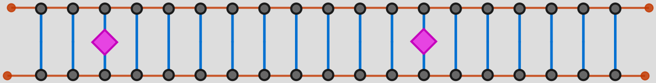
MPS in graphical notation

$$|A, L, R\rangle = \sum_{i_1,i_2,\cdots,i_n} \langle \mathrm{L}|A_{i_1}(1)A_{i_2}(2)\cdots A_{i_n}(n)|\mathrm{R}\rangle\; |i_1,\cdots,i_n\rangle$$

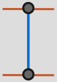
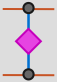
Notation: $[A_i]_{\mathbf{k},\mathbf{l}} =$  and k ——— $l = \sum \delta_{k,l}$ gives:

$|A, L, R\rangle =$ 

Example: computation of correlations

$\langle A|\mathcal{O}(i_k)\mathcal{O}(i_\ell)|A\rangle =$ 

can be done efficiently by iterating 2 maps:

$\Phi =$  and $\Phi_{\mathcal{O}} =$ 

Continuous Matrix Product States

Type of ansatz for bosons on a fine grained lattice

- Matrices $A_{i_k}(x)$ where the index i_k corresponds to $\psi^{\dagger i_k}(x)|0\rangle$ in physical space.

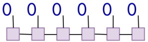
Informal cMPS definition

$$A_0 = \mathbb{1} + \varepsilon Q$$
$$A_1 = \varepsilon R$$
$$A_2 = \frac{(\varepsilon R)^2}{\sqrt{2}}$$
$$\dots$$
$$A_n = \frac{(\varepsilon R)^n}{\sqrt{n}}$$

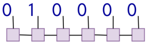
so we go from ∞ to 2 matrices

Fixed by:

- Finite particle number



$\propto \quad 1$



$\propto \quad \varepsilon$

- Consistency

Continuous Matrix Product States

Introduced by Verstraete and Cirac in 2010

Definition

$$|Q,R,\omega\rangle = \langle\omega_L|\mathcal{P}\exp\left\{\int_0^L dx\; Q\otimes\mathbb{1} + R\otimes\psi^\dagger(x)\right\}|\omega_R\rangle\;|0\rangle_\psi$$

- Q, R are $D\times D$ matrices,
- $|\omega_L\rangle$ and $|\omega_R\rangle$ are boundary vectors $\in\mathbb{C}^D$, for p.b.c. $\langle\omega_L|\cdot|\omega_R\rangle \to \mathrm{tr}[\,\cdot\,]$
- $[\psi(x),\psi^\dagger(y)] = \delta(x-y)$

Idea: A generalized coherent state

Computations

Some correlation functions

$$\left\langle \hat{\psi}(x)^\dagger \hat{\psi}(x) \right\rangle = \operatorname{Tr}\left[e^{TL} (R \otimes \overline{R}) \right]$$

$$\left\langle \hat{\psi}(x)^\dagger \hat{\psi}(0)^\dagger \hat{\psi}(0) \hat{\psi}(x) \right\rangle = \operatorname{Tr}\left[e^{T(L-x)} (R \otimes \overline{R}) e^{Tx} (R \otimes \overline{R}) \right]$$

$$\left\langle \hat{\psi}(x)^\dagger \left[-\frac{d^2}{dx^2} \right] \hat{\psi}(x) \right\rangle = \operatorname{Tr}\left[e^{TL} ([Q, R] \otimes [\overline{Q}, \overline{R}]) \right]$$

with $T = Q \otimes \mathbb{1} + \mathbb{1} \otimes \bar{Q} + R \otimes \bar{R}$

Example

Lieb-Liniger Hamiltonian

$$\mathcal{H} = \int_{-\infty}^{+\infty} dx \left[\frac{d\hat{\psi}^\dagger}{dx} \frac{d\hat{\psi}}{dx} - \mu \hat{\psi}^\dagger \hat{\psi} + c \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \right]$$

Solve by **minimizing**: $\langle Q, R | \mathcal{H} | Q, R \rangle = f(Q, R)$

Standard CMPS and ϕ^4

Applying cMPS to the ϕ^4 Hamiltonian

$$\langle Q, R|\widehat{h}_{\phi^4}|Q, R\rangle = \infty$$

Oh no!

The short distance behavior of cMPS is the wrong one, even the free theory is hard to approximate.

Going relativistic

Infusing some “high-energy” knowledge into tensor networks

Towards relativistic CMPS

Local basis in position of the QFT: $\psi^\dagger$, ϕ, π, $|0\rangle_\psi$
Diagonal basis of the free part: $a_k^\dagger$, $|0\rangle_a$

Bogoliubov transform

Go from $\hat{\psi}(x), \hat{\psi}^\dagger(x)$ to $a(p), a^\dagger(p)$ with

$$a(p) = \frac{1}{\sqrt{2}}\left(\sqrt{\omega_p}\,\hat{\phi}(p) + \frac{\hat{\pi}(p)}{\sqrt{\omega_p}}\right) \quad \text{with} \quad \omega_p = \sqrt{p^2 + m^2}$$

which yields

$$H_0 = \int \mathrm{d}p\, \omega_p\, \frac{1}{2}\left(a_p^\dagger a_p + a_p a_p^\dagger\right)$$

Go from $|0\rangle_\psi$ to $|0\rangle_a$
and
Go from $\psi(x)$ to $a(x) = \int \mathrm{d}p\; a(p) \mathrm{e}^{ipx} \neq \psi(x)$

Relativistic CMPS

Definition

$$|R, Q\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q \otimes \mathbb{1} + R \otimes a^\dagger(x)\right]\right\}|0\rangle_a$$

Some properties

1. $|0, 0\rangle = |0\rangle_a$ is the ground state of H_0 hence exact CFT UV fixed point (because interaction super-renormalizable)
2. $\langle Q, R|h_{\phi^4}|Q, R\rangle$ is finite for all Q, R (not trivial)

Consequence on the Hamiltonian

Hamiltonian density in $a(x)$ basis

H is local in $\psi(x)$, not in $a(x)$...

$$
\begin{aligned}
H = & \int \mathrm{d}x_1 \mathrm{d}x_2 D(x_1 - x_2) a^\dagger(x_1) a(x_2) \\
& + \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 K(x_1, x_2, x_3, x_4) a(x_1) a(x_2) a(x_3) a(x_4) + 4a^\dagger aaa + 3a^\dagger a^\dagger aa \\
& + \text{h.c.}
\end{aligned}
$$

But fortunately exponentially decreasing: K is horrible, but decays $\propto e^{-m|x|}$.

The nightmarish optimization

Compute $e_0 = \langle Q, R | h_{\phi^4} | Q, R \rangle$ and $\nabla_{Q,R} e_0$

1. Contains an algebraic part identical to standard cMPS
2. Involves horrible quadruple integrals without analytic solutions

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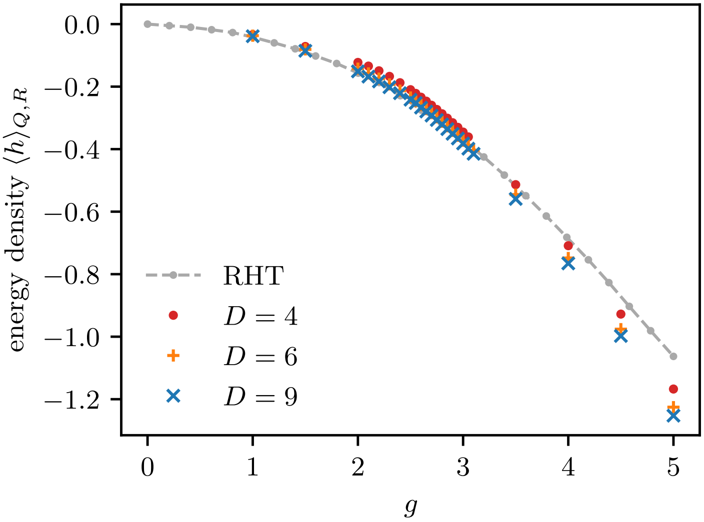
One needs to do TDVP (i.e. variational optimization with a metric). Equivalent with imaginary time evolution with large time-step.

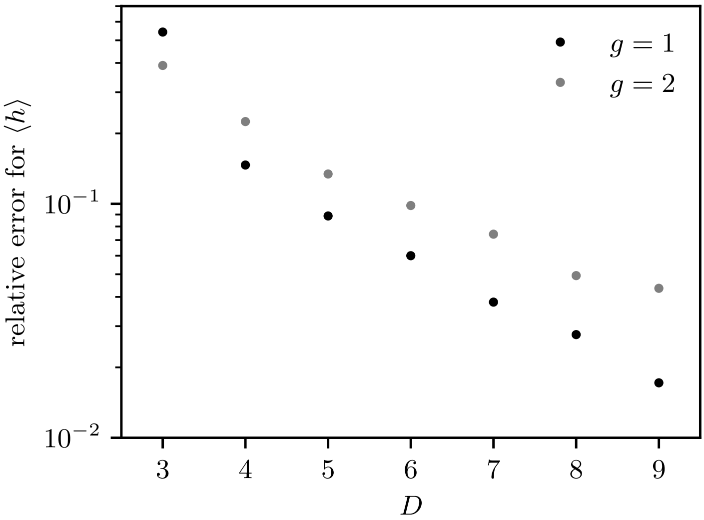
Results and discussion

Results



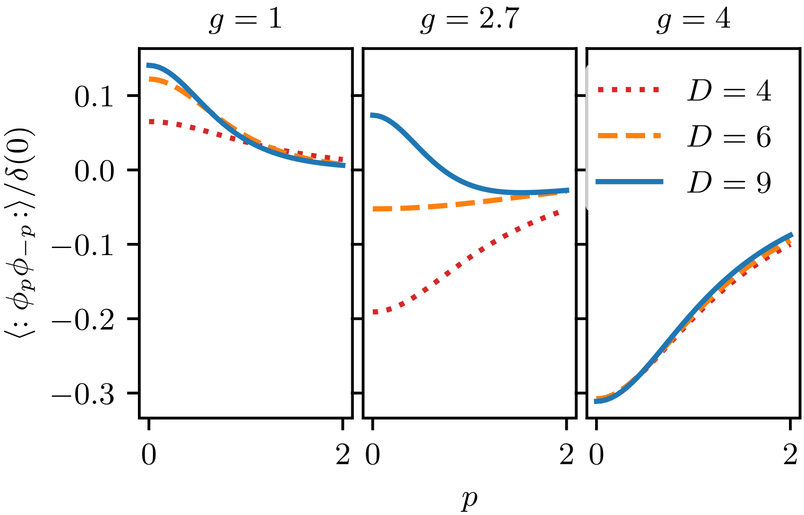
Compared with the Renormalized Hamiltonian Truncation results of Rychkov and Vitale from 2015.

Results



Compared with the "high precision" Renormalized Hamiltonian Truncation results of Elias Miro, Rychkov, and Vitale from 2017 for $g = 1$ and $g = 2$

Results



Normal ordered momentum two point function $\langle : \phi_p \phi_{-p} : \rangle_{Q,R}$

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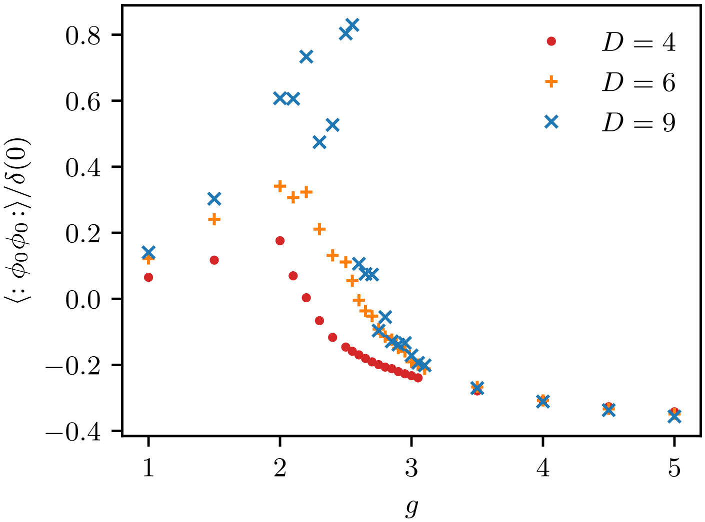

Normal ordered momentum two point function at zero momentum $\langle : \phi_0\phi_0 : \rangle_{Q,R}$

Comparison with renormalized Hamiltonian truncation

Ren. Hamiltonian truncation

IR cutoff L, energy truncation E_T

- Uses a vector space
- Function to minimize is quadratic, hence linear problem
- Number of parameters $\propto e^{L \times E_T}$
- Error $\propto 1/E_T^3$
- Spectrum easy

Relativistic CMPS

entanglement truncation D

- Uses a manifold
- Minimization is a priori non-trivial but doable
- Number of parameters $\propto D^2$
- Error $o(1/D^\alpha)$, $\forall\ \alpha$ (folklore)
- Fixed t correl. functions easy

Note: real world not asymptotic. RCMPS has expensive prefactors, and RHT can use reliable extrapolations

Extensions

- To other bosonic theories in $1+1$ with poly $V(\phi)$ $\rightarrow$ easy
- To fermionic theories in $1+1$ $\rightarrow$ feasible
- To $2+1$ and $3+1$ dimensions $\rightarrow$ very difficult
  (lattice tensor networks will probably rule in $2+1$ and $3+1$ for numerics)

Summary

1. New ansatz for $1 + 1$ relativistic QFT
2. No cutoff, UV or IR (a first?)
3. UV is captured exactly even at $D = 0$
4. Efficient (cost poly D, error superpoly $1/D$)
5. Rigorous (variational)