Continuous tensor network states

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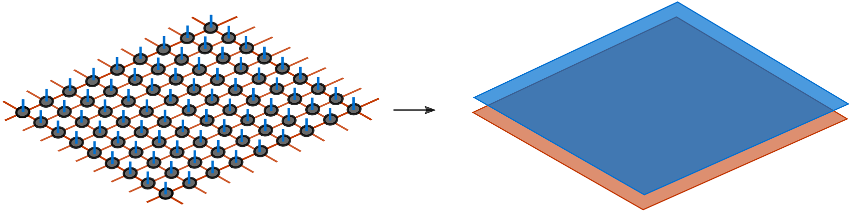
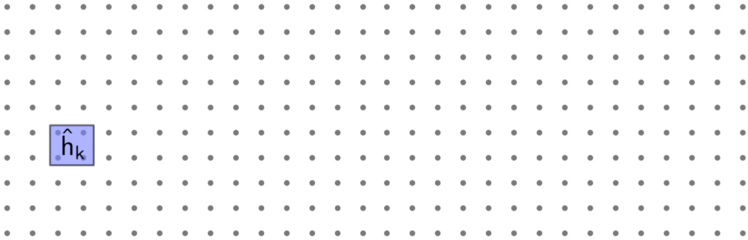


European Tensor Network
ICCUB School 2021
October 1st, 2021

Quantum many-body problem on the lattice

Typical condensed matter problem: $|\psi\rangle = \sum c_{i_1, i_2, \cdots, i_n} |i_1, i_2, \cdots i_N\rangle$



Problem:

Finding the low energy states of

$$\hat{H} = \sum_{k=1}^{N} \hat{h}_k$$

is **hard** because $\dim \mathscr{H} \propto 2^N$ for spins

Possible solutions

- Perturbation theory
  but weak coupling
- Monte Carlo
  but imprecise and sign problem
- **Compression** $2^N \to N^\alpha$
  with controllable error

The direct compression approach

Variational method for ground state search

1. Guess a manifold $\mathcal{M} \subset \mathscr{H}$ with few parameters ν *i.e.* $\dim \mathcal{M} \ll \dim \mathscr{H}$
2. Tune ν to minimize energy $\nu = \operatorname{argmin}_{\nu \in \mathcal{M}} \frac{\langle \nu | H | \nu \rangle}{\langle \nu | \nu \rangle}$ and get $|\text{ground state}\rangle \simeq |\nu\rangle$

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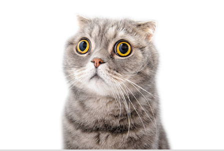
Reason for compression (classical)



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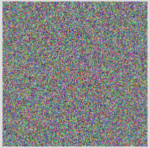

"typical" image

atypical $\Longrightarrow$ compressible

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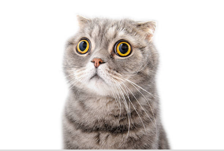

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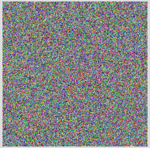

"typical" image

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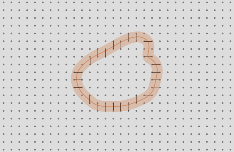
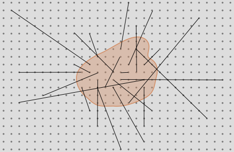
Reason for compression (quantum)



low energy state

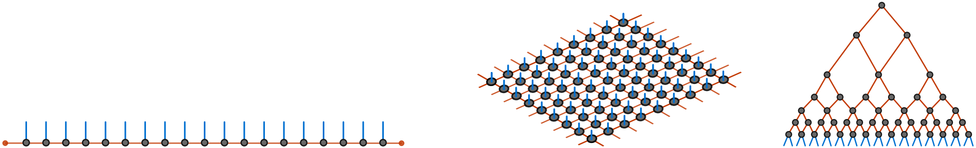


random state

area law $=$ atypical $\Longrightarrow$ compressible

Tensor network states in a nutshell

.zip or **.jpg** for complex quantum states that appear in Nature

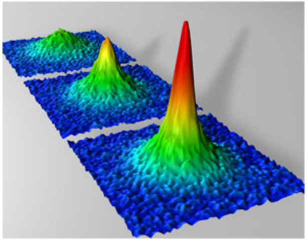


1. **Exponential reduction:** $2^N \longrightarrow N \times D^{2d}$ parameters
   [N number of spins, D amount of entanglement, d space dimension $(1,2,3)$]

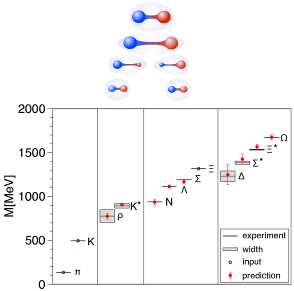
2. **Efficient compression:** compression error $\leqslant e^{-D}$ or $1/\text{superpoly}(D)$
   [For a large number of *a priori* non-trivial problems]

Many non-trivial problems are continuous

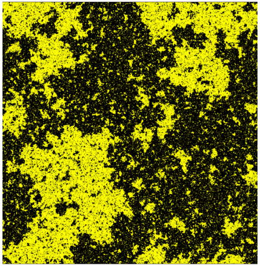
Non-relativistic QFT
including quantum gases and fractional quantum Hall phases



Relativistic QFT
including, ultimately, quantum chromodynamics

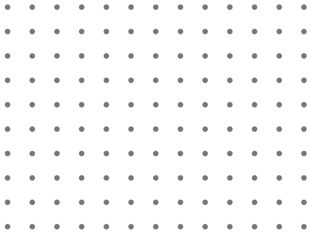


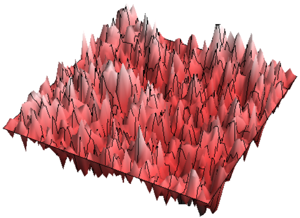
Critical systems
classical and quantum at 2nd order phase transitions



The quantum many-body problem in the continuum

From the lattice to the continuum and Quantum Field Theory (QFT)





$$|\Psi\rangle = \sum_{i_1, i_2, \cdots, i_N} c_{i_1 i_2 \cdots i_N} |i_1 i_2 \cdots i_N\rangle \quad \longrightarrow \quad |\Psi\rangle = \int \mathcal{D}\phi \; \psi(\phi) \, |\phi\rangle$$

New problem: 2^N $\mathbb{C}$-parameters $\rightarrow$ $\dim \mathscr{H} = \infty^\infty$ even at finite size!

Question Can one compress ∞^∞ down to a manageable number of parameters?
$\rightarrow$ Feynman argued it was impossible in a 1987 conference

Feynman's criticism

Difficulties in Applying the Variational Principle to Quantum Field Theories[1]

so I tried to do something along these lines with quantum chromodynamics. So I'm talking on the subject of the application of the variational principle to field theoretic problems, but in particular to quantum chromodynamics.

I'm going to give away what I want to say, which is that I didn't get anywhere! I got very discouraged and I think I can see why the variational principle is not very useful. So I want to take, for the sake of argument, a very strong view – which is stronger than I really believe – and argue that it is no damn good at all!

Feynman's requirement in a nutshell

1. **Extensive parameterization**
   Number of parameters $\propto L^{\alpha}$ at most for system size L
2. **Computable expectation values**
   ψ known $\implies \langle \mathcal{O}(x)\mathcal{O}(y)\rangle_{\psi}$ computable
3. **Not oversensitive to the UV**
   no runaway minimization where higher and higher momenta get fitted

Numerical continuum limit

Change the model so you can apply known methods

1. **Discretize**

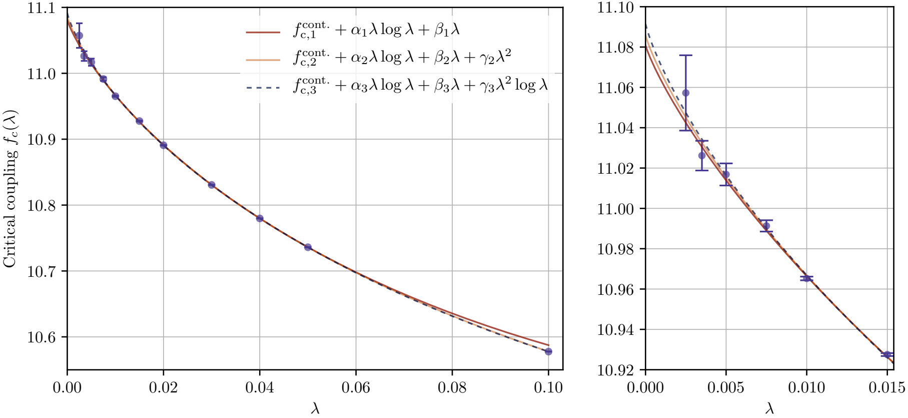
   State: $|\Psi\rangle = \int \mathcal{D}\phi \; \psi(\phi) \, |\phi\rangle \quad \longrightarrow \quad |\Psi_\varepsilon\rangle = \sum_{i_1, i_2, \cdots, i_N} c_{i_1 i_2 \cdots i_N} |i_1 i_2 \cdots i_N\rangle$

   Hamiltonian: $H = \int \mathrm{d}x \, h(x) \quad \longrightarrow \quad H_\varepsilon = \sum_i h_i$

2. **Solve** with tensor networks for fixed lattice spacing
3. **Extrapolate** to zero lattice spacing

Numerical continuum limit

Critical coupling for ϕ_2^4 lattice field theory as a function of lattice spacing λ

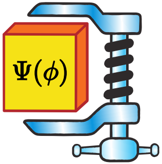


Error grows as $\lambda \to 0$

Working directly in the continuum

Big challenge

Compress field wavefunctions $\psi(\phi)$ and use them to solve the continuous-many-body problem directly



Status– since Feynman, breakthrough in 2010 and recent progress

	non-relativistic	relativistic	critical
$d = 1$ space	Verstraete-Cirac 2010	2021	
$d \geqslant 2$ space	2019		

no idea	heuristics	clear definition	algorithm

Outline

1. Continuous Matrix Product States
   $\rightarrow$ on the board - first half
2. (Relativistic) Continuous Matrix Product States
3. Continuous tensor networks in $d \geqslant 2$

Relativistic continuous matrix product states

RCMPS: *A variational ansatz to solve $1+1d$ relativitic QFT without discretization or cutoff and to arbitrary precision*

Relativistic continuous matrix product states

RCMPS: *A variational ansatz to solve $1+1d$ relativitic QFT without discretization or cutoff and to arbitrary precision*

Two papers

- Variational method in relativisitic QFT without cutoff (short)
  `arXiv:2102.07733v2`
- Relativistic continuous matrix product states for QFT without cutoff (long)
  `arXiv:2102.07741v2`

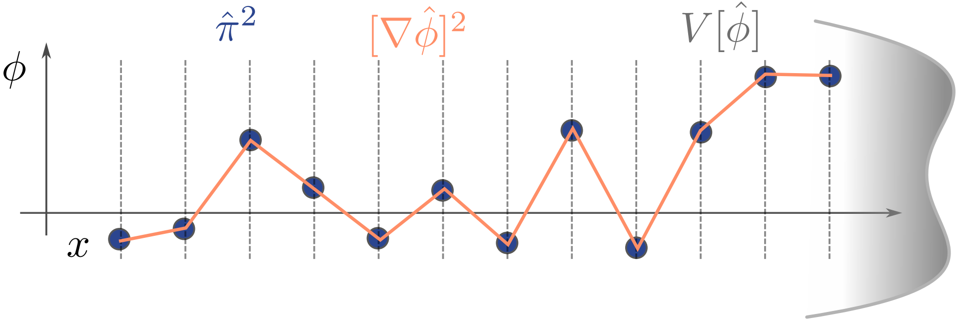
Outline for relativistic QFT in 1+1

1. Scalar fields in $1+1$ dimensions
2. Variational method in the continuum
3. Relativistic twist $\psi \to a$ for CMPS
4. Numerics (and how to achieve $D^6 \to D^3$)
5. Open questions

Basics of relativistic scalar field theory

from a condensed matter viewpoint

Intuitive definition: canonical quantization



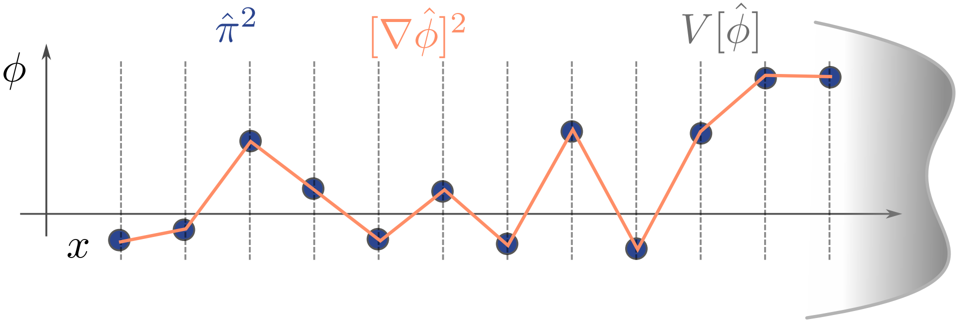
Hamiltonian

A continuum of nearest neighbor coupled anharmonic oscillators

$$\hat{H} = \int_{\mathbb{R}^d} \mathrm{d}^d x \quad \underbrace{\frac{\hat{\pi}(x)^2}{2}}_{\text{on-site inertia}} + \underbrace{\frac{[\nabla\hat{\phi}(x)]^2}{2}}_{\text{spatial stiffness}} + \underbrace{V(\hat{\phi}(x))}_{\text{on-site potential}}$$

with canonical commutation relations $[\hat{\phi}(x), \hat{\pi}(y)] = i\delta^d(x-y)\mathbb{1}$ (*i.e.* bosons)

Intuitive definition



Hilbert space

Fock space $\mathscr{H}_{\mathrm{QFT}} = \mathscr{F}[L^2(\mathbb{R}^d)]$ – just like $x, p \to (a, a^\dagger)$ do $\hat{\pi}, \hat{\phi} \to \hat{\psi}, \hat{\psi}^\dagger$

$$|\Psi\rangle = \sum_{n=0}^{+\infty} \int \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \underbrace{\varphi_n(x_1, x_2, \cdots, x_n)}_{\text{wave function}} \underbrace{\hat{\psi}^\dagger(x_1)\hat{\psi}^\dagger(x_2)\cdots\hat{\psi}^\dagger(x_n)}_{\text{local oscillator creation}} |\mathrm{vac}\rangle$$

What are the problems compared to non-relativistic field theories

The Hamiltonian is ill defined on all states in the Hilbert space because of infinite zero point energy *i.e.* terms $\propto \hat{\psi}(x)\hat{\psi}^\dagger(x)$

$$\langle \Psi_1 | \hat{H} | \Psi_2 \rangle = \pm\infty \quad \text{and even} \quad \langle \text{vac} | \hat{H} | \text{vac} \rangle \propto \delta^d(0) = +\infty$$

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If the divergent vacuum terms are removed, the Hamiltonian is not bounded from below

$$\forall \ |\Psi\rangle \in \mathscr{H}, \ \langle \Psi | \hat{H}_{\text{finite}} | \Psi \rangle = \text{finite but} \ \ \exists \ \Psi_n \ \text{s.t.} \lim_{n\to+\infty} \langle \Psi_n | H_{\text{finite}} | \Psi_n \rangle = -\infty$$

How are they are solved in the free case - Hamiltonian

Bogoliubov transform

Go from $\hat{\psi}(x), \hat{\psi}^\dagger(x)$ to $a(p), a^\dagger(p)$ with

$$a(p) = \frac{1}{\sqrt{2}} \left(\sqrt{\omega_p}\, \hat{\phi}(p) + i \frac{\hat{\pi}(p)}{\sqrt{\omega_p}} \right) \quad \text{with} \quad \omega_p = \sqrt{p^2 + m^2}$$

which yields

$$H_0 = \int \mathrm{d}p\, \omega_p\, \frac{1}{2} \left(a_p^\dagger a_p + a_p a_p^\dagger \right)$$

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Solution

- Take $H_{\text{QFT}} \equiv \,:H:_a$
- $|\text{free ground state}\rangle = |\text{vacuum}\rangle_a$
- $\mathscr{H}$ built from $a_{p_1}^\dagger \cdots a_{p_n}^\dagger |\text{vacuum}\rangle_a$

This solves the problematic free part exactly, and allows to define a finite interaction (in $1+1$)

Example: rigorous operator definition of ϕ_2^4

Renormalized ϕ_2^4 theory

$$H = \int \mathrm{d}x \, \frac{:\pi^2:_a}{2} + \frac{:(\nabla\phi)^2:_a}{2} + \frac{m^2}{2} :\phi^2:_a + g :\phi^4:_a$$

(note that $:\diamondsuit:_a$ depends on m)

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1. Rigorously defined relativistic QFT without cutoff (Wightman QFT)
2. Vacuum energy density finite
3. Very difficult to solve unless $g \ll m^2$ (perturbation theory)
4. Phase transition around $f_c = \frac{g}{4m^2} = 11$ i.e. $g \simeq 2.7$ in mass units

Hilbert spaces of RQFT in $1+1$

Two operator basis

The $\psi^\dagger(x)$ basis

Local oscillator basis

- $+$ Local in ϕ, π
- $+$ Natural for discretization
- $-$ Divergent and ill-defined

The $a_k^\dagger$ basis

"Relativistic" oscillator basis

- $-$ Non-local
- $-$ Less natural for discretization
- $+$ Regular and well-defined

The variational method
in the continuum

The variational method

In the Hamiltonian formulation:

- Guess a **finite dimensional submanifold** $\mathscr{M}$ of the QFT Hilbert space $\mathscr{H}$
- Find the ground state by minimizing $\langle H \rangle$:

$$|\text{ground}\rangle \simeq |\psi\rangle = \underset{\mathscr{M}}{\text{argmin}} \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}$$

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Example: naive Hamiltonian truncation

With an IR cutoff, momenta are discrete. Take as submanifold $\mathscr{M}$ the **vector space** spanned by:

$$a^\dagger_{k_1} a^\dagger_{k_2} \cdots a^\dagger_{k_r} |0\rangle_a$$

where $r \leqslant r_{\text{max}}$ and $k \leqslant k_{\text{max}}$ (one possible truncation)

Feynman's objection

Feynman's requirement for variational wavefunctions in RQFT (1987)

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All methods so far break one at least:

- Hamiltonian truncation fails at 1 (but works fairly well through its renormalized refinements)
- Tensor networks succeed at 1 and 2 but fail (a priori) at 3

Continuous matrix product states

Continuous Matrix Product States

Introduced by Verstraete and Cirac in 2010

Definition

$$|Q, R\rangle = \mathrm{tr}\left[\mathcal{P}\exp\left\{\int_0^L dx\; Q \otimes \mathbb{1} + R \otimes \psi^\dagger(x)\right\}\right] |0\rangle_\psi$$

- Q, R are $D \times D$ matrices,
- The trace is taken over this matrix space
- $[\psi(x), \psi^\dagger(y)] = \delta(x - y)$
- $\psi^\dagger(x)$ is non-relativistic creation operator (i.e. $\phi(x) = \frac{1}{\sqrt{2\nu}}[\psi(x) + \psi^\dagger(x)]$)
- $|0\rangle_\psi$ is the associated Fock vacuum

Idea:

- From MPS: a continuum limit
- From QFT: a sort of generalized "non-commutative" coherent state

Computations

Some correlation functions

$$\left\langle \hat{\psi}(x)^\dagger \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{TL} (R \otimes \overline{R}) \right]$$

$$\left\langle \hat{\psi}(x)^\dagger \hat{\psi}(0)^\dagger \hat{\psi}(0) \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{T(L-x)} (R \otimes \overline{R}) e^{Tx} (R \otimes \overline{R}) \right]$$

$$\left\langle \hat{\psi}(x)^\dagger \left[-\frac{d^2}{dx^2} \right] \hat{\psi}(x) \right\rangle = \mathrm{Tr}\left[e^{TL} ([Q, R] \otimes [\overline{Q}, \overline{R}]) \right]$$

with $T = Q \otimes \mathbb{1} + \mathbb{1} \otimes \bar{Q} + R \otimes \bar{R}$

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Example

Lieb-Liniger Hamiltonian

$$\mathcal{H} = \int_{-\infty}^{+\infty} dx \left[\frac{d\hat{\psi}^\dagger}{dx} \frac{d\hat{\psi}}{dx} - \mu \hat{\psi}^\dagger \hat{\psi} + c \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \right]$$

Solve by **minimizing**: $\langle Q, R | \mathcal{H} | Q, R \rangle = f(Q, R)$

State of the art on CMPS

Contrary to common beliefs, CMPS are fairly efficient

1. Fully variational calculations at $D = 256$ by Ganahl-Rincon-Vidal 2016
2. Recently Tuybens-De Nardis-Haegeman-Verstraete `arXiv:2006.01801` included open-boundaries efficiently

Standard CMPS and relativistic fields

Applying cMPS to e.g. the ϕ^4 Hamiltonian

$$\langle Q, R|\widehat{h}_{\phi^4}|Q, R\rangle = \infty$$

Oh no!

The short distance behavior of cMPS is the wrong one, even the free theory is hard to approximate.

A possible fix by Haegeman-Cirac-Osborne-Verschelde-Verstraete 2010:

$$H \to H_\Lambda := H + \frac{1}{\Lambda^2}\int \mathrm{d}x\, \frac{(\partial_x \pi)^2}{2}$$

Going relativistic

Changing of operator basis

Towards relativistic CMPS

Local basis in position of the QFT: $\psi^\dagger$, ϕ, π, $|0\rangle_\psi$
Diagonal basis of the free part: $a_k^\dagger$, $|0\rangle_a$

Bogoliubov transform

Go from $\hat{\psi}(x), \hat{\psi}^\dagger(x)$ to $a(p), a^\dagger(p)$ with

$$a(p) = \frac{1}{\sqrt{2}}\left(\sqrt{\omega_p}\,\hat{\phi}(p) + i\frac{\hat{\pi}(p)}{\sqrt{\omega_p}}\right) \quad \text{with} \quad \omega_p = \sqrt{p^2+m^2}$$

which yields

$$H_0 = \int \mathrm{d}p\, \omega_p \, \frac{1}{2}\left(a_p^\dagger a_p + a_p a_p^\dagger\right)$$

Go from $|0\rangle_\psi$ to $|0\rangle_a$
and
Go from $\psi(x)$ to $a(x) = \int \mathrm{d}p\; a(p)\mathrm{e}^{ipx} \neq \psi(x)$

Relativistic CMPS

Definition

$$|R, Q\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q\otimes \mathbb{1} + R\otimes a^\dagger(x)\right]\right\}|0\rangle_a$$

Some properties

1. $|0,0\rangle = |0\rangle_a$ is the ground state of H_0 hence exact CFT UV fixed point (because interaction super-renormalizable)
2. $\langle Q, R|h_{\phi^4}|Q, R\rangle$ is finite for all Q, R (not trivial)

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$a(x)$ is not covariant but the state cannot be exactly Poincaré invariant anyway!

Consequence on the Hamiltonian

Hamiltonian density in $a(x)$ basis

H is local in $\psi(x)$, not in $a(x)$...

$$H = \int \mathrm{d}x_1 \mathrm{d}x_2 D(x_1 - x_2) a^\dagger(x_1) a(x_2)$$
$$+ \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 K(x_1, x_2, x_3, x_4) a(x_1) a(x_2) a(x_3) a(x_4) + 4a^\dagger aaa + 3a^\dagger a^\dagger aa$$
$$+ \text{h.c.}$$

But fortunately exponentially decreasing: K decays $\propto e^{-m|x|}$ for $|x| \gg m$.

The variational algorithm

Procedure:
Compute $e_0 = \langle Q, R | h_{\phi^4} | Q, R \rangle$ and $\nabla_{Q,R} e_0$
Minimize e_0 with **TDVP** aka gradient descent with a metric

Computations of e_0 and ∇e_0 in a nutshell:

1. Contains an algebraic part identical to standard cMPS
2. Involves quadruple integrals without analytic solutions
   Initial v1 idea: compute the integrals with Quadpack $\rightarrow$ cost D^6

Computing vertex operators

Main insight

$\langle : e^{b\phi(x)} : \rangle_{QR}$ computable by solving an ODE with cost $\propto D^3$

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$\langle :e^{b\phi(x)}: \rangle_{QR}$ computable by solving an ODE with cost $\propto D^3$

Going from $\phi(x)$ to $a(x)$ gives:

$$\begin{aligned} \langle :e^{b\phi(0)}: \rangle_{QR} &= \left\langle \exp\left[b\int J(x)a^\dagger(x)\right]\exp\left[b\int J(x)a(x)\right]\right\rangle_{Q,R} \\ &= Z_{bJ,bJ} \end{aligned} \tag{1}$$

with

$$J(x) = \int \mathrm{d}k \frac{1}{\sqrt{2\omega_k}} e^{ikx} \tag{2}$$

and Z_{j_1,j_2} is just the generating functional

$$Z_{j_1,j_2} = \mathrm{tr}\left[\mathcal{P}\exp\int \mathbb{T} + j_1(x)R\otimes\mathbb{1} + +j_2(x)\mathbb{1}\otimes\bar{R}\,\mathrm{d}x\right] \tag{3}$$

Algorithm v2 $\propto D^3$

1. Compute $Z_{bJ,bJ}$ by solving the ODE

$$\partial_x \rho = \mathcal{L}\rho + bJ(x)(R\rho + \rho R^\dagger)$$

and taking the trace at $x = +\infty$

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Bottom line

Solve with cost $\propto D^3$ all theories with $V(\phi)$ poly $:\phi^n:$ or exponential $:e^{b\phi}:$ (including Sine/Sinh-Gordon and thus Fermionic theories via bosonization)

Scaling comparison with renormalized Hamiltonian truncation

Ren. Hamiltonian truncation

IR cutoff L, energy truncation E_T

- Uses a vector space
- Function to minimize is quadratic, hence linear problem
- Number of parameters $\propto \mathrm{e}^{L \times E_T}$
- Error $\propto 1/E_T^3$

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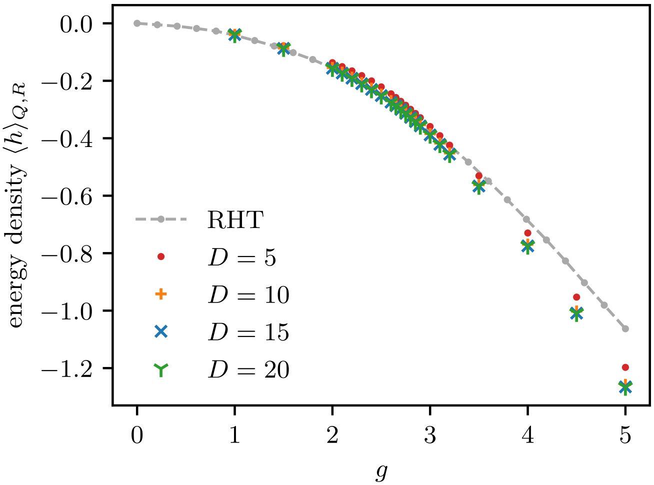
Relativistic CMPS

entanglement truncation D

- Uses a manifold
- Minimization is a priori non-trivial but doable
- Number of parameters $\propto D^2$
- Error $o(1/D^\alpha)$, $\forall\ \alpha$ (folklore)

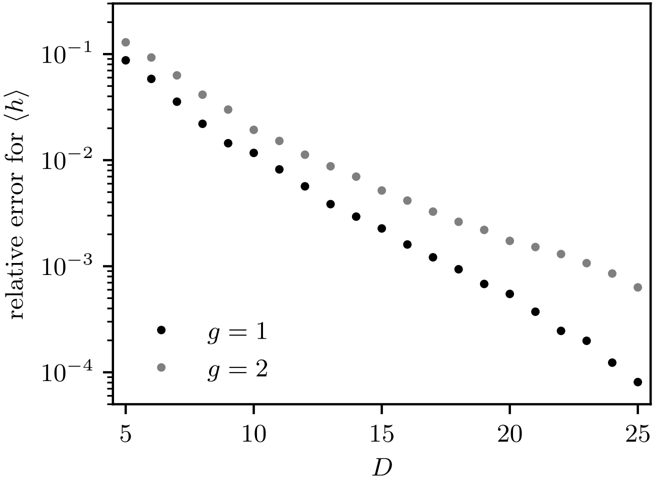
Results

Energy density



Results

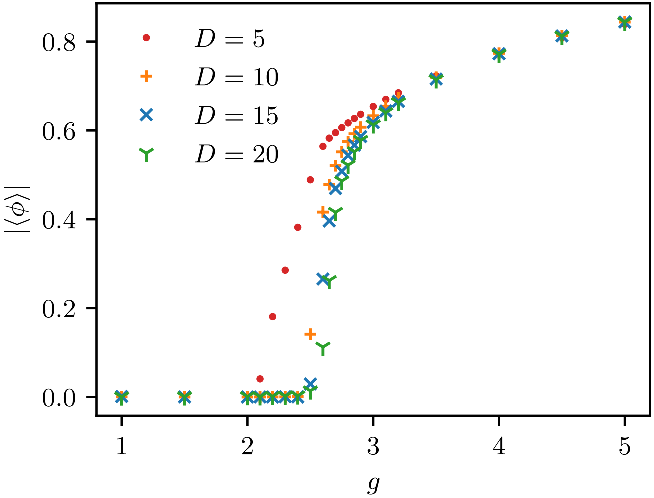
Error in energy density



Approximately exact value extrapolated from $D = 32$ (boostrapped error $< 10^{-4}$). More precise than high precision RHT. Pushable to $D > 40$

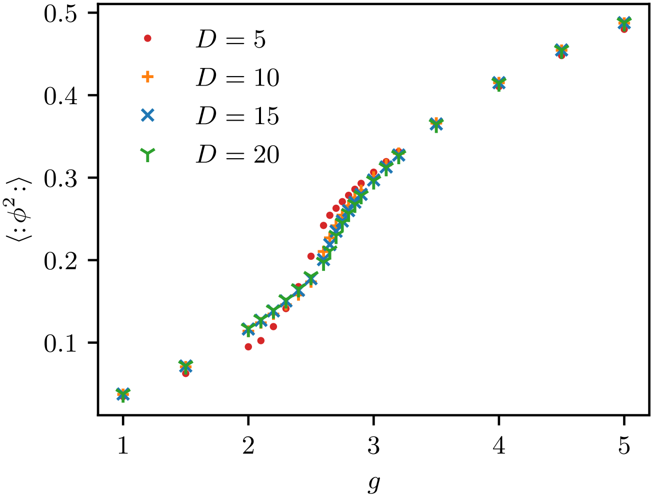
Results

Magnetization $\langle\phi\rangle$



Results

$\langle : \phi^2 : \rangle$



Open problems and perspectives

New entanglement entropy

Conjecture

For the notion of space locality is induced by $a^\dagger(x), a(x)$ (instead of usual $\phi(x)$), gapped QFT ground states verify the area law with a **finite** prefactor.

- This entanglement entropy is weird from a relativistic point of view
- But captures the notion of approximability with tensor network states

Useful notion? Can the conjecture be proved?

More general short distance behavior

RCMPS have the short distance behavior of a free CFT (fairly generic in HEP)

Can one deal with relevant perturbations of other UV CFTs (e.g. Ising)?

Equivalent of $a(x)$? Coulomb gas construction?

Higher dimensions

RQFT difficulty

Normal ordering / tadpole cancellation no longer sufficient
Whightman QFT still have Hilbert space, but less explicit (not free Fock space)

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(non-relativistic) Tensor network difficulty

Continuous tensor network states less developed in $2+1$

1. Proposal with Ignacio Cirac: R, Q promoted to fields, needed to preserve Euclidean invariance
2. Successfully tested on Gaussian problems with Teresa Karanikolaou (also independently in Ghent by Bastiaan Aelbrecht)
3. Need to solve a boundary $1+1$ RQFT to compute more general expectation values

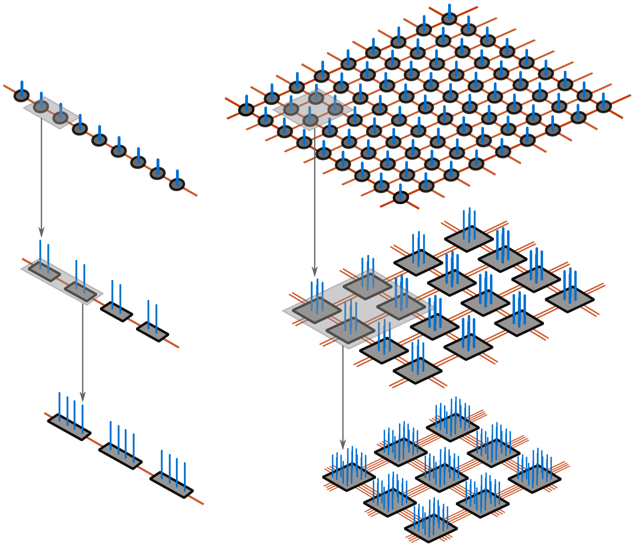
Non-relativistic $2+1$ now seems feasible thanks to RCMPS...

Summary of relativistic CMPS

1. Ansatz for $1+1$ relativistic QFT
2. No cutoff, UV or IR
3. UV is captured exactly even at $D=0$
4. Efficient (cost poly D, error at most superpoly $1/D$) and now competitive
5. Rigorous (variational)

What about $d \geqslant 2$

Continuous Tensor Networks: blocking

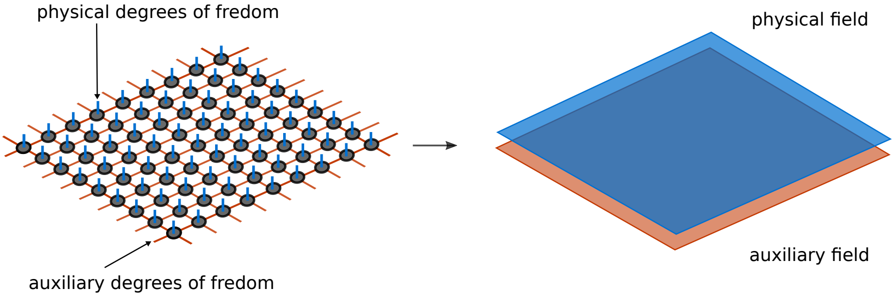


Upon **blocking**:

- The **physical** Hilbert space dimension increases
- The **bond** (auxiliary space) dimension D increases too

Now from bottom to top, fine graining will yield zero bond dimension.

Result



AT, J. I. Cirac, *2019*

Continuous tensor network state (heuristically)

State $|V,\alpha\rangle$ of $d+1$ QFT from an auxiliary d dimensional theory of random fields ϕ:

$$|V,\alpha\rangle = \int \mathcal{D}\phi \; \exp\left\{ -\int \mathrm{d}^d x \; \mathcal{L}_V[\phi(x)] - \alpha[\phi(x)] \underbrace{\hat{\psi}^\dagger(x)}_{\text{creation}} \right\} |\Omega\rangle$$

Choice of tensor around which to expand...

For **MPS**, not much choice:

 $=$ $+ \; \varepsilon \cdots$

$$= \mathbb{1} \otimes |0\rangle + \varepsilon \, Q \otimes |0\rangle + \varepsilon \, R \otimes \psi^{\dagger}(x)|0\rangle$$

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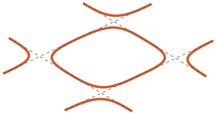
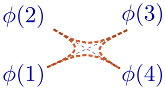
3. Take the sum of pairs of identities in both directions
$T^{(0)} =$  $+$ 



Ansatz

1 – Take a "Trivial" tensor:

$$T^{(0)}_{\phi(1),\phi(2),\phi(3),\phi(4)} =$$



$$\sim \exp\left\{ \frac{-1}{2} \sum_{k=1}^{D} [\phi_k(1) - \phi_k(2)]^2 + [\phi_k(2) - \phi_k(3)]^2 + [\phi_k(3) - \phi_k(4)]^2 + [\phi_k(4) - \phi_k(1)]^2 \right\}$$

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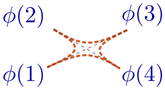

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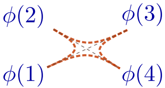

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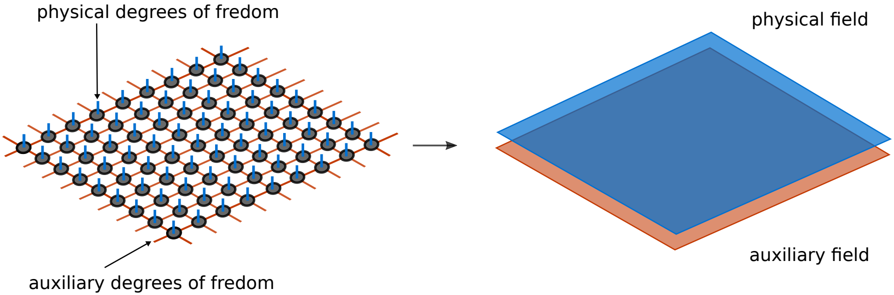
3 – Realize tensor contraction = functional integral and trivial tensor gives free field measure.

Result



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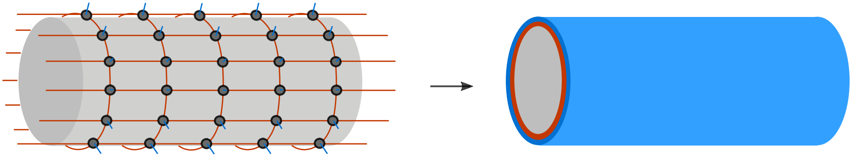
Operator definition



$$|V,\alpha\rangle =$$

$$\mathrm{tr}\left[\mathcal{T}\exp\left(-\int_0^T \mathrm{d}\tau \int_S \mathrm{d}x\, \frac{\hat{\pi}_k(x)\hat{\pi}_k(x)}{2} + \frac{\nabla\hat{\phi}_k(x)\nabla\hat{\phi}_k(x)}{2} + V[\hat{\phi}(x)] - \alpha[\hat{\phi}(x)]\,\psi^{\dagger}(\tau,x)\right)\right]$$

where:

- $\hat{\phi}_k(x)$ and $\hat{\pi}_k(x)$ are D independent canonically conjugated pairs of (auxiliary) field *operators*: $[\hat{\phi}_k(x), \hat{\phi}_l(y)] = 0$, $[\hat{\pi}(x)_k, \hat{\pi}_l(y)] = 0$, and $[\hat{\phi}_k(x), \hat{\pi}_l(y)] = i\delta_{k,l}\,\delta(x-y)$ acting on a space of $d-1$ dimensions.

Wave-function definition

A generic state $|\Psi\rangle$ in Fock space can be written:

$$|\Psi\rangle = \sum_{n=0}^{+\infty} \int_{\Omega^n} \frac{\varphi_n(x_1, \cdots, x_n)}{n!} \psi^\dagger(x_1) \cdots \psi^\dagger(x_n) |0\rangle$$

where φ_n is a symmetric n-particle wave-function

Functional integral representation

$$\varphi_n(x_1, \cdots, x_n) = \langle\, \alpha[\phi(x_1)] \cdots \alpha[\phi(x_n)] \,\rangle_{\text{aux}}$$

with:

$$\langle \cdot \rangle_{\text{aux}} = \int \mathcal{D}\phi \;\cdot\; B(\phi|_{\partial\Omega}) \exp\left[-\frac{1}{2}\int_{\Omega} \mathrm{d}^d x \,[\nabla\phi_k(x)]^2 + V[\phi(x)]\right]$$

- $\sim$ Ansatz wave-function for Quantum Hall, but CFT $\rightarrow$ QFT

Expressivity and stability

How big are cTNS?

Stability

The sum of two cTNS of bond field dimension D_1 and D_2 is a cTNS with bond field dimension $D \leqslant D_1 + D_2 + 1$:

$$|V_1, \alpha_1\rangle + |V_2, \alpha_2\rangle = |W, \beta\rangle$$

Expressiveness

All states in the Fock space can be approximated by cTNS:

- A field coherent state is a cTNS with $D = 1$
- Stability allows to get all sums of field coherent states

Computations

Define generating functional for normal ordered correlation functions

$$\mathcal{Z}_{j',j} = \frac{1}{\langle V, \alpha | V, \alpha \rangle} \langle V, \alpha | \exp\left(\int dx\, j'(x) \psi^\dagger(x) \right) \exp\left(\int dx\, j(x) \psi(x) \right) |V, \alpha\rangle$$

Operator representation

$$\mathcal{Z}_{j'j} = \mathrm{tr}\left[B \otimes B^* \mathcal{T} \exp\left\{ \int_{-T/2}^{T/2} \left(\mathrm{T}_{j'j} - \int_S j \cdot j' \right) \right\} \right]$$

with **transfer matrix**:

$$\mathrm{T}_{j'j} = \int_S dx\, \mathcal{H}(x) \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{H}^*(x) + \left(\alpha[\hat{\phi}(x)] + j'(x) \right) \otimes \left(\alpha[\hat{\phi}(x)]^* + j(x) \right)$$

and $\mathcal{H}(x) = \sum_{k=1}^{D} \frac{[\hat{\pi}_k(\mathbf{x})]^2 + \left[\nabla \hat{\phi}_k(\mathbf{x})\right]^2}{2} + V[\hat{\phi}(x)]$

$\Longrightarrow$ cMPS brought us from 1 to 0, cTNS bring us from d to $d-1$.

Contraction

- In general, need boundary relativistic CMPS to contract
- If $V(\phi) = V^{(0)}\phi_i V_{ij}^{(2)}\phi_j$
  and $\alpha(\phi) = \alpha^{(0)} + \alpha_i^{(1)}\phi_i$
  Gaussian $\rightarrow$ exactly contractible

Example:

$$H = \int \nabla\hat{\psi}^\dagger \nabla\hat{\psi} + \mu\psi^\dagger\hat{\psi} - \lambda(\hat{\psi}^\dagger\hat{\psi}^\dagger + \hat{\psi}\hat{\psi})$$

Gaussian example

Work done by Teresa Karanikolaou with help from Patrick Emonts (PRR 2021)

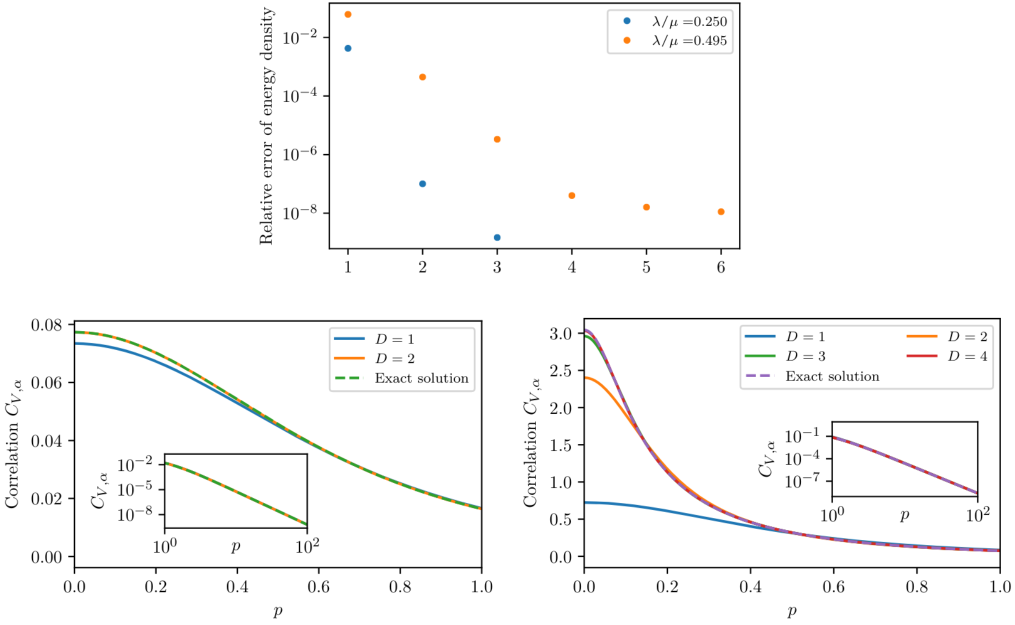
$$H = \int_{\mathbb{R}^2} \nabla\hat{\psi}^\dagger \nabla\hat{\psi} + \mu\psi^\dagger\hat{\psi} - \lambda(\hat{\psi}^\dagger\hat{\psi}^\dagger + \hat{\psi}\hat{\psi})$$

in $d = 2$ energy density $\langle h \rangle$ divergent, but CTNS also divergent!

$$\langle h \rangle = e_0^r + \log(\Lambda) e_0^\infty \tag{4}$$

1. Analytically minimize the divergent part
2. Numerically minimize the remain finite (renormalized part)

Energy and correlation functions



Summary of CTNS

$$|V, B, \alpha\rangle = \int \mathcal{D}\phi \, \exp\left\{ -\int_\Omega \mathrm{d}^d x \, \frac{1}{2} \sum_{k=1}^{D} [\nabla \phi_k(x)]^2 + V[\phi(x)] - \alpha[\phi(x)] \, \psi^\dagger(x) \right\} |0\rangle$$

Continuous tensor network states are natural continuum limits of tensor network states and natural higher d extensions of continuous matrix product states.

1. Obtained from discrete tensor networks
2. Can be made Euclidean invariant
3. **Motto of tensor networks**: trade a dimension for a variational optimization
4. Still needs to be used to approximate non-trivial non-Gaussian ground states