The variational method for relativistic fields

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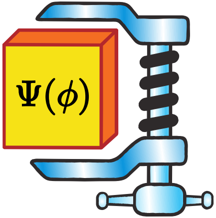
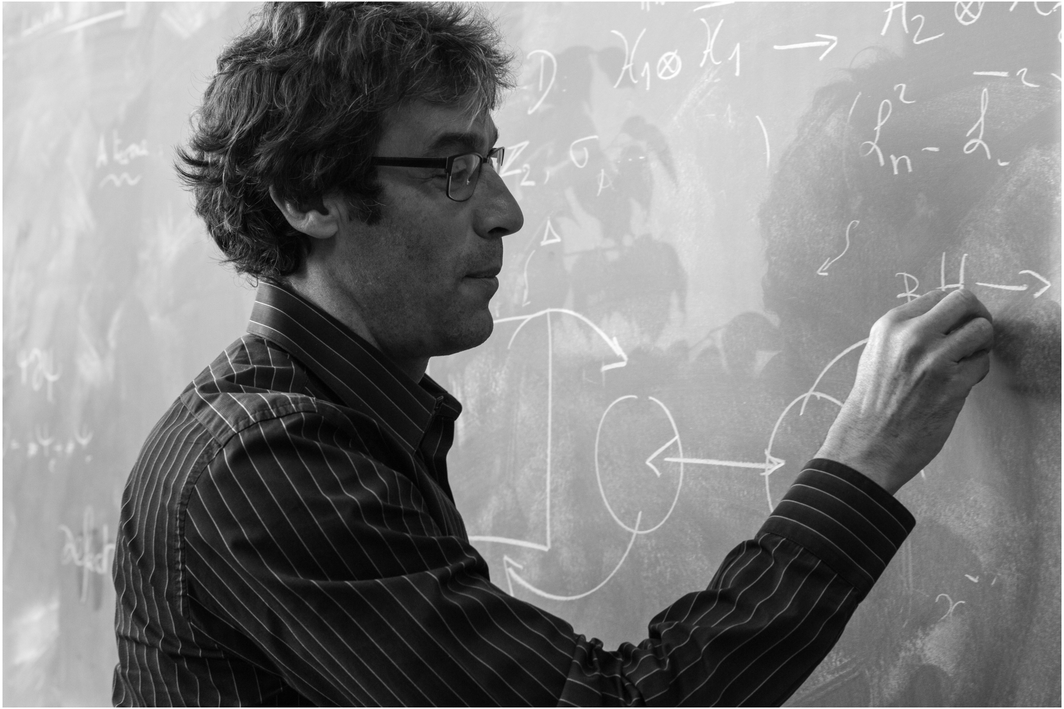


Mathematical Harmony and the quantum world
for (and by) amateurs
October 15th, 2021



Strategies to beat interacting quantum field theory

Two ways to attack *real world* quantum field theories non-perturbatively

1. Start **simpler** so that it becomes **simpler** [*e.g.* ϕ^4_2]
2. Start **more complex** so that it becomes **simpler** [*e.g.* $\mathcal{N} = 4$ *SYM*]

ϕ^4_2 - pile of dirt

QCD - Everest

$\mathcal{N} = 4$ *SYM* - Chrysler building

more on this on `tilloy.wordpress.com`

An example of well defined field theory

Renormalized ϕ_2^4 theory

$$H = \int \mathrm{d}x \, \frac{:\pi^2:_a}{2} + \frac{:(\nabla\phi)^2:_a}{2} + \frac{m^2}{2} :\phi^2:_a + g :\phi^4:_a$$

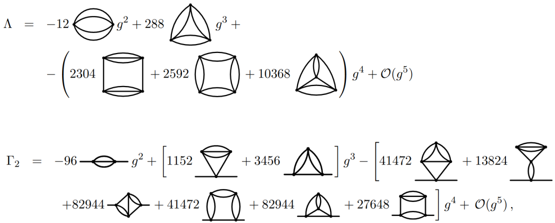
(note that $:\diamondsuit:_a$ depends on m)

1. Rigorously defined relativistic QFT without cutoff (Wightman QFT)
2. Vacuum energy density finite
3. Very difficult to solve unless $g \ll m^2$ (perturbation theory)
4. Phase transition around $f_c = \frac{g}{4m^2} = 11$ i.e. $g \simeq 2.7$ in mass units

Two (main) games in town

Perturbation theory

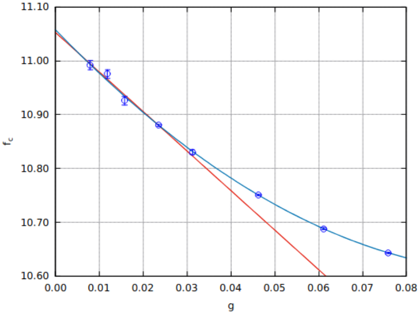
+ resummation



state of the art is $O(g^8)$

arXiv:1805.05882
Serone, Spada, Villadoro

Lattice Monte-Carlo



arXiv:1807.03381
Bronzin, De Palma, Guagnelli

The direct compression approach

Variational method for ground state search

1. Guess a manifold $\mathcal{M} \subset \mathscr{H}$ with few parameters ν *i.e.* $\dim \mathcal{M} \ll \dim \mathscr{H}$
2. Tune ν to minimize energy $\nu = \operatorname{argmin}_{\nu \in \mathcal{M}} \frac{\langle \nu | H | \nu \rangle}{\langle \nu | \nu \rangle}$ and get $|\text{ground state}\rangle \simeq |\nu\rangle$

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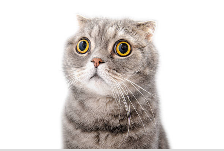
Reason for compression (classical)



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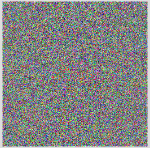

"typical" image

atypical $\Longrightarrow$ compressible

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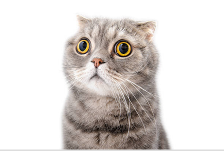

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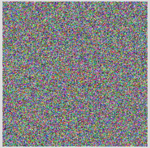

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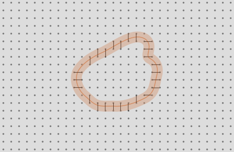
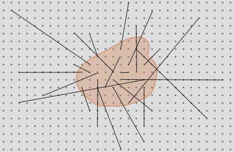
Reason for compression (quantum)



low energy state



random state

area law $=$ atypical $\Longrightarrow$ compressible

Feynman's criticism

Difficulties in Applying the Variational Principle to Quantum Field Theories[1]

so I tried to do something along these lines with quantum chromodynamics. So I'm talking on the subject of the application of the variational principle to field theoretic problems, but in particular to quantum chromodynamics.

I'm going to give away what I want to say, which is that I didn't get anywhere! I got very discouraged and I think I can see why the variational principle is not very useful. So I want to take, for the sake of argument, a very strong view – which is stronger than I really believe – and argue that it is no damn good at all!

Feynman's requirement in a nutshell

1. **Extensive parameterization**
   Number of parameters $\propto L^{\alpha}$ at most for system size L (not $\propto \mathrm{e}^{L}$)
2. **Computable expectation values**
   ψ known $\implies \langle \mathcal{O}(x)\mathcal{O}(y)\rangle_{\psi}$ computable
3. **Not oversensitive to the UV**
   no runaway minimization where higher and higher momenta get fitted

Elegantly swallowing the bullet

Example: naive Hamiltonian truncation

With an IR cutoff L, momenta are discrete. Take as submanifold $\mathscr{M}$ the **vector space** spanned by:

$$|k_1, k_2, \cdots, k_r\rangle = a^\dagger_{k_1} a^\dagger_{k_2} \cdots a^\dagger_{k_r} |0\rangle_a$$

such that $\langle k_1 k_2 \cdots k_r | H | k_1 k_2 \cdots k_r \rangle \leqslant E_{\text{trunc}} \;\; \rightarrow$ finite dimensional

Breaks **extensiveness**

- number of parameters $\propto \mathrm{e}^{L \times E_{\text{trunc}}}$
- error $\propto E_{\text{trunc}}^{-3}$ (with renormalization refinements)

still good results, see Rychkov & Vitale `arXiv:1412.3460`

Relativistic continuous matrix product states

RCMPS: *A variational ansatz to solve* $1+1d$ *relativitic QFT without discretization or cutoff and to arbitrary precision*

Definition

(Verstraete & Cirac 2010 for non-relativistic $\longrightarrow$ AT 2021 for relativistic)

A RCMPS is a manifold of states parameterized by 2 $(D \times D)$ matrices Q, R

$$|Q, R\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q \otimes \mathbb{1} + R \otimes a^{\dagger}(x)\right]\right\}|0\rangle_a$$

with

- $a(x) = \frac{1}{2\pi}\int \mathrm{d}k\, \mathrm{e}^{ikx} a_k$ where $a_k = \frac{1}{\sqrt{2}}\left(\sqrt{\omega_p}\,\hat{\phi}(p) + i\frac{\hat{\pi}(p)}{\sqrt{\omega_p}}\right)$
- trace taken over $\mathbb{C}^D$
- $\mathcal{P}$ path-ordering exponential

Basic properties of RCMPS

$$|Q,R\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q\otimes \mathbb{1} + R\otimes a^\dagger(x)\right]\right\}|0\rangle_a$$

Feynman's checklist:

1. **Extensive** because of $\mathcal{P}\exp\int$
2. Obervables **computable** at cost D^3 (non trivial!)
   requires $[a(x), a^\dagger(y)] = \delta(x-y)$ i.e. *quantum noise* techniques
3. **No UV problems**
   $|0,0\rangle = |0\rangle_a$ is the ground state of H_0 hence exact CFT UV fixed point
   $\langle Q,R|h_{\phi^4}|Q,R\rangle$ is finite for all Q,R (not trivial!)

The variational algorithm

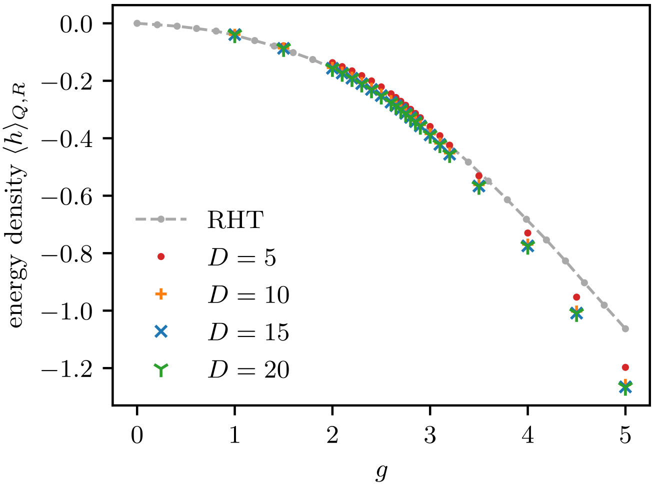
Procedure:
Compute $e_0 = \langle Q, R | h_{\phi^4} | Q, R \rangle$ and $\nabla_{Q,R} e_0$
Minimize e_0 with **TDVP** aka gradient descent with a metric

Computations of e_0 and ∇e_0 in a nutshell:

1. $V_b = \langle : e^{b\phi(x)} : \rangle_{QR}$ computable by solving an ODE with cost $\propto D^3$
2. $\langle : \phi^n : \rangle_{QR}$ computable doing $\partial_b^n V_b \Big|_{b=0} \quad \rightarrow \quad \propto D^3$
3. $e_0 = \langle h \rangle_{QR}$ computable by summing such terms at cost $D^3 \quad \rightarrow \quad \propto D^3$
4. ∇e_0 computable by solving the adjoint ODE (backpropagation) $\quad \rightarrow \quad \propto D^3$

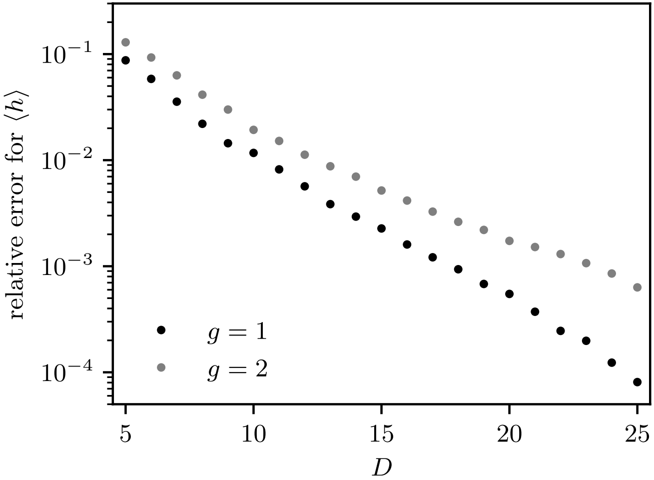
Results

Energy density



Results

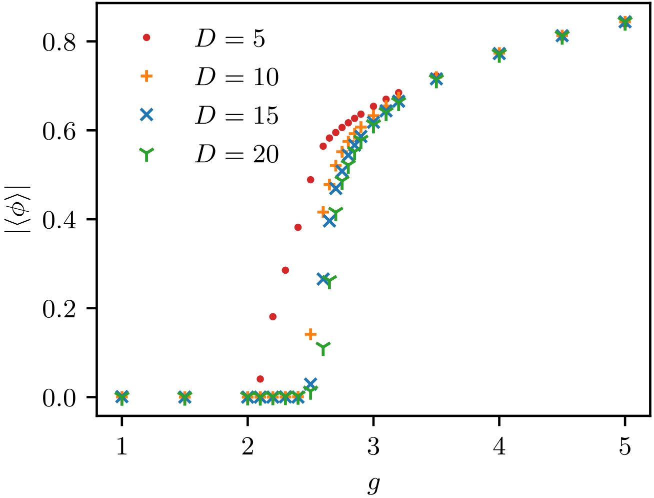
Error in energy density



Approximately exact value extrapolated from $D = 32$ (boostrapped error $< 10^{-4}$). More precise than high precision RHT. Pushable to $D > 40$

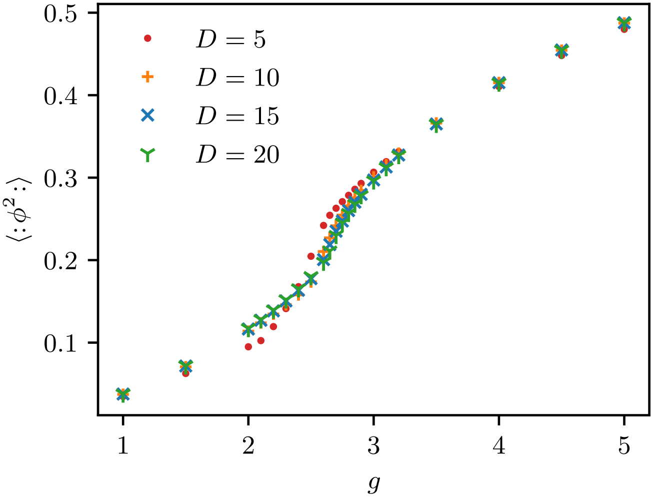
Results

Magnetization $\langle\phi\rangle$



Results

$\langle : \phi^2 : \rangle$



Scaling comparison

Ren. Hamiltonian truncation

IR cutoff L, energy truncation E_T

- Uses a vector space
- Function to minimize is quadratic, hence linear problem
- Number of parameters $\propto \mathrm{e}^{L \times E_T}$
- Error $\propto 1/E_T^3$

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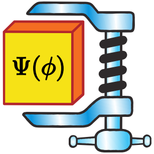
Relativistic CMPS

entanglement truncation D

- Uses a manifold
- Minimization is a priori non-trivial but doable
- Number of parameters $\propto D^2$
- Error $o(1/D^\alpha)$, $\forall\ \alpha$ (folklore)

Summary

$$|Q, R\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q \otimes \mathbb{1} + R \otimes a^{\dagger}(x)\right]\right\}|0\rangle_a$$



1. Ansatz for $1+1$ relativistic QFT
2. No cutoff, UV or IR, extensive, computable
3. UV is captured exactly even at $D = 0$
4. Efficient (cost poly D, error 1/superpoly D) and now competitive
5. Rigorous (variational)