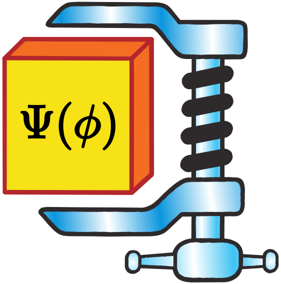
Attacking the Sinh-Gordon model with relativistic continuous matrix product states



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ICTP / SISSA joint seminar

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Inria

Quantum field theory: general objective

Long term goal

Find methods to solve "real world" quantum field theories (even without structure) to good (machine?) precision

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2. Lattice Monte Carlo $\leftarrow$ need discretization / slow convergence of error / sign

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2. Lattice Monte Carlo ← need discretization / slow convergence of error / sign

3 **promising alternatives**

1. Bootstrap / SDP relaxations / Sum of Squares
2. Renormalization group ← functional or tensor network RG
3. **Variational method** ← Hamiltonian truncation or tensor network states

Variational method and RCMPS

In $1+1$ dimensions, relativistic continuous matrix product states are an ansatz with few parameters to efficiently find ground states and compute observables [arXiv:2102.07733 and arXiv:2102.07741]

$$|Q,R\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q\otimes \mathbb{1} + R\otimes a^\dagger(x)\right]\right\}|0\rangle_a$$

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Useful next steps: extend to fermions, gauge theories, $2+1$ and $3+1$ dim

What I did: look at the Sinh-Gordon model because it is weird and controversial

The Sinh-Gordon model

An exactly solvable model that is surprisingly subtle. Two recent studies

- Könik, Lájer, and Mussardo [KLM] arXiv:2007.00154
- Bernard and LeClair [BLC] arXiv:2112.05490

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[Equal-time quantization] Hamiltonian definition

$$H_{\mathrm{ShG}}(\beta) = \int \mathrm{d}x \, \frac{:\pi^2:_m}{2} + \frac{:(\nabla\phi)^2:_m}{2} + \frac{m^2}{\beta^2} :\cosh(\beta\phi):_m$$

[Radial quantization] Dilation operator definition

$$D_{\mathrm{ShG}}(b) = D_0 + \mu \int_C \mathrm{d}z \, [\mathcal{V}_b(z, z^*) + \mathcal{V}_{-b}(z, z^*)]$$

Equivalent formulations with $b = \beta/\sqrt{8\pi}$ and $\mu = \frac{m^{2+2b^2}}{2^{4+2b^2}\pi b^2} e^{2b^2\gamma_E}$

The Sinh-Gordon model: puzzles

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Should be easy:

1. Intuitively should always make sense ($\cosh(\beta\phi)$ always relevant)
2. S-matrix, energy density, masses, vertex operators, "exactly" known
3. Apparent $b \to b^{-1}$ duality with normalized coupling $b = \beta/\sqrt{8\pi}$

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But unclear what the domain of validity of the formula is...

- Mass vanishes at $b = 1$ and likely stays at 0 [KLM and BLC]
- Likely no self-duality
- Could the exact formula break down before $b = 1$?
- Very hard to check numerically (despite thorough exploration of KLM)

Outline

1. The variational method in the continuum
2. Relativistic continuous matrix product states (RCMPS)
3. Warm-up with ϕ_2^4 and $\cos(\beta\phi)$
4. $\cosh(\beta\phi)$ numerics
5. Some lessons

The variational method
in the continuum

The direct compression approach

Variational method for ground state search

1. Guess a manifold $\mathcal{M} \subset \mathscr{H}$ with few parameters ν *i.e.* $\dim \mathcal{M} \ll \dim \mathscr{H}$
2. Tune ν to minimize energy $\nu = \operatorname{argmin}_{\nu \in \mathcal{M}} \frac{\langle \nu | H | \nu \rangle}{\langle \nu | \nu \rangle}$ and get $|\text{ground state}\rangle \simeq |\nu\rangle$

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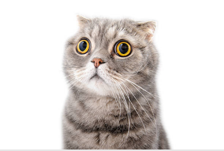
Reason for compression (classical)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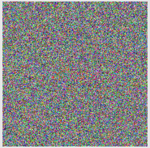
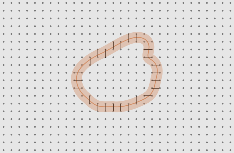
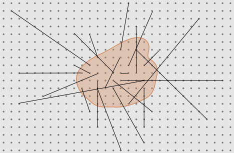

"typical" image

atypical $\implies$ compressible

Reason for compression (quantum)


low energy state


random state

area law $=$ atypical $\implies$ compressible

Feynman's criticism

Difficulties in Applying the Variational Principle to Quantum Field Theories[1]

so I tried to do something along these lines with quantum chromodynamics. So I'm talking on the subject of the application of the variational principle to field theoretic problems, but in particular to quantum chromodynamics.

I'm going to give away what I want to say, which is that I didn't get anywhere! I got very discouraged and I think I can see why the variational principle is not very useful. So I want to take, for the sake of argument, a very strong view – which is stronger than I really believe – and argue that it is no damn good at all!

Feynman's requirement in a nutshell

1. **Extensive parameterization**
   Number of parameters $\propto L^{\alpha}$ at most for system size L (not $\propto \mathrm{e}^{L}$)
2. **Computable expectation values**
   ψ known $\implies \langle \mathcal{O}(x)\mathcal{O}(y) \rangle_{\psi}$ computable
3. **Not oversensitive to the UV**
   no runaway minimization where higher and higher momenta get fitted

Elegantly swallowing the bullet

Example: naive Hamiltonian truncation

With an IR cutoff L, momenta are discrete. Take as submanifold $\mathscr{M}$ the **vector space** spanned by:

$$|k_1, k_2, \cdots, k_r\rangle = a^\dagger_{k_1} a^\dagger_{k_2} \cdots a^\dagger_{k_r} |0\rangle_a$$

such that $\langle k_1 k_2 \cdots k_r | H | k_1 k_2 \cdots k_r \rangle \leqslant E_{\text{trunc}} \rightarrow$ finite dimensional

Breaks **extensiveness**

- number of parameters $\propto \mathrm{e}^{L \times E_{\text{trunc}}}$
- error $\propto E_{\text{trunc}}^{-3}$ (with renormalization refinements)

still good results, see *e.g.* Rychkov & Vitale for ϕ^4_2 `arXiv:1412.3460`

Intuition

1- Extensive parameterization and 2- Computable expectation values

Realized by **tensor network states** on the lattice
e.g. in $1+1$ dimensions: Matrix Product states (MPS)

$$|\psi(A)\rangle := \sum_{i_1, i_2, \dots, i_N} \mathrm{tr}\left[A_{i_1} A_{i_2} \cdots A_{i_N}\right] |i_1, i_2, \dots, i_N\rangle$$

where A_i are matrices $\in \mathcal{M}_D(\mathbb{C})$

3- Not oversensitive to the UV

Realized by **Hamiltonian truncation**, *i.e.* working in the Fock basis

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Strategy: MPS $\underset{\text{continuum limit}}{\longrightarrow}$ CMPS (2010) $\underset{\text{change of basis}}{\longrightarrow}$ RCMPS (2021)

Relativistic continuous matrix product states (RCMPS)

Relativistic continuous matrix product states

RCMPS: *A variational ansatz to solve $1+1d$ relativitic QFT without discretization or cutoff and to (in principle) arbitrary precision*

Definition

RCMPSs are a manifold of states parameterized by 2 $(D \times D)$ matrices Q, R

$$|Q, R\rangle = \text{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q \otimes \mathbb{1} + R \otimes a^\dagger(x)\right]\right\}|0\rangle_a$$

with

- $a(x) = \frac{1}{2\pi}\int \mathrm{d}k\, \mathrm{e}^{ikx} a_k$ where $a_k = \frac{1}{\sqrt{2}}\left(\sqrt{\omega_k}\,\hat{\phi}(k) + i\frac{\hat{\pi}(k)}{\sqrt{\omega_k}}\right)$
- trace taken over $\mathbb{C}^D$
- $\mathcal{P}$ path-ordering exponential

Basic properties of RCMPS

$$|Q, R\rangle = \text{tr}\left\{\mathcal{P}\exp\left[\int \text{d}x\, Q\otimes \mathbb{1} + R\otimes a^\dagger(x)\right]\right\}|0\rangle_a$$

Feynman's checklist:

1. **Extensive** because of $\mathcal{P}\exp\int$
2. Obervables **computable** at cost D^3 (non trivial!)
   requires $[a(x), a^\dagger(y)] = \delta(x-y)$
3. **No UV problems**
   $|0,0\rangle = |0\rangle_a$ is the ground state of H_0 hence exact CFT UV fixed point
   $\langle Q, R| :P(\phi): |Q, R\rangle$ is finite for all Q, R (not trivial!)

The variational algorithm

Procedure:
Compute $e_0 = \langle Q, R|h|Q, R\rangle$ and $\nabla_{Q,R}e_0$
Minimize e_0 with **TDVP** (essentially Riemannian gradient descent)

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Computations of e_0 and ∇e_0 in a nutshell:

1. $V_b = \langle :e^{b\phi(x)}:\rangle_{QR}$ computable by solving an ODE with cost $\propto D^3$
2. $\langle :\phi^n:\rangle_{QR}$ computable doing $\partial_b^n V_b\Big|_{b=0}$ $\rightarrow \propto D^3$
3. $e_0 = \langle h\rangle_{QR}$ computable by summing such terms at cost D^3 $\rightarrow \propto D^3$
4. ∇e_0 computable by solving the adjoint ODE (backpropagation) $\rightarrow \propto D^3$

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Functioning `Julia` implementation. `OptimKit.jl` to solve the Riemannian minimization, `KrylovKit.jl` to solve fixed point equations, `DifferentialEquations.jl` (`Vern7` solver) to solve ODE. Soon `Rcmps.jl`?

Warmup with ϕ_2^4 and $\cos(\beta\phi)$

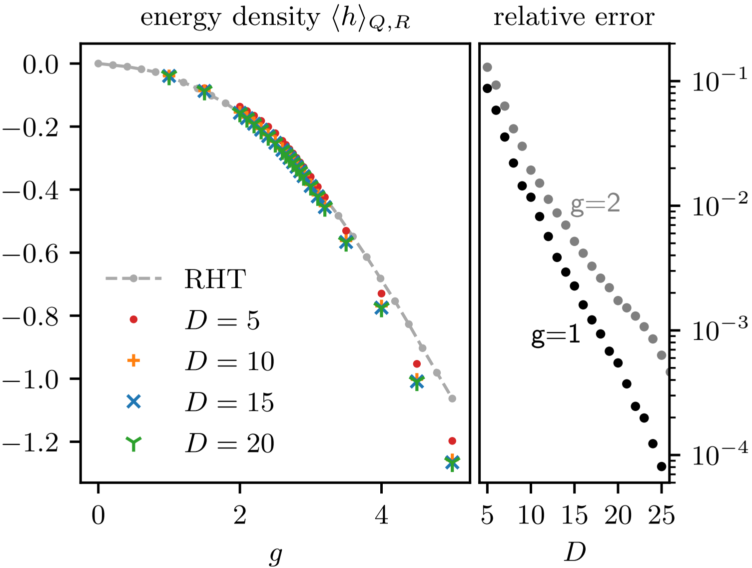
Hamiltonian definition of ϕ_2^4

Renormalized ϕ_2^4 theory

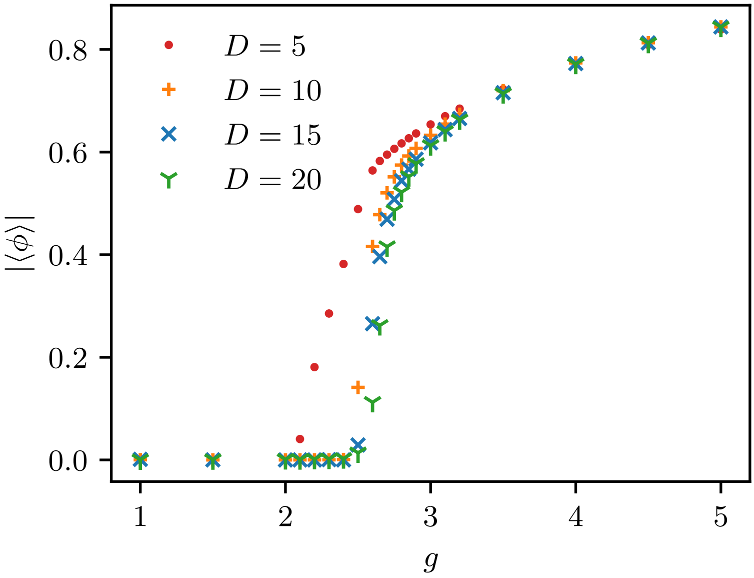
$$H = \int \mathrm{d}x \, \frac{:\pi^2:_m}{2} + \frac{:(\nabla\phi)^2:_m}{2} + \frac{m^2}{2} :\phi^2:_m + g :\phi^4:_m$$

1. Rigorously defined relativistic QFT without cutoff (Wightman QFT)
2. Vacuum energy ε_0 density finite
3. Difficult to solve unless $g \ll m^2$ (perturbation theory)
4. Phase transition around $f_c = \frac{g}{4m^2} = 11$ i.e. $g \simeq 2.7$ in mass units

Results: ϕ_2^4 energy density



Results: ϕ_2^4 – field expectation value $\langle\phi\rangle$



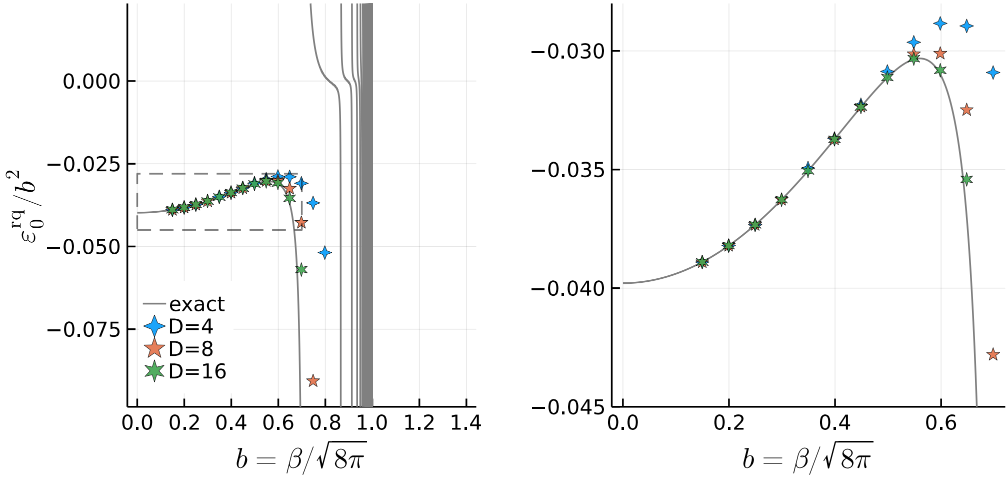
Hamiltonian definition of Sine-Gordon theory

Renormalized $\cos(\beta\phi)$ theory

$$H = \int \mathrm{d}x \, \frac{:\pi^2:_m}{2} + \frac{:(\nabla\phi)^2:_m}{2} - \frac{m^2}{\beta^2} :\cos(\beta\phi):_m$$

1. Well defined for $b = \beta/\sqrt{8\pi} < 1/\sqrt{2}$
2. Ground energy density $\to -\infty$ for $b \to 1/\sqrt{2}$ but renormalizable until $b = 1$
3. Vertex operators, mass spectrum, and (renormalized) energy known exactly

Results: $\cos(\beta\phi)$ (rescaled) energy density



Fits arbitrarily well for $b \in [0, 1/\sqrt{2}[$, collapses to $-\infty$ for b larger

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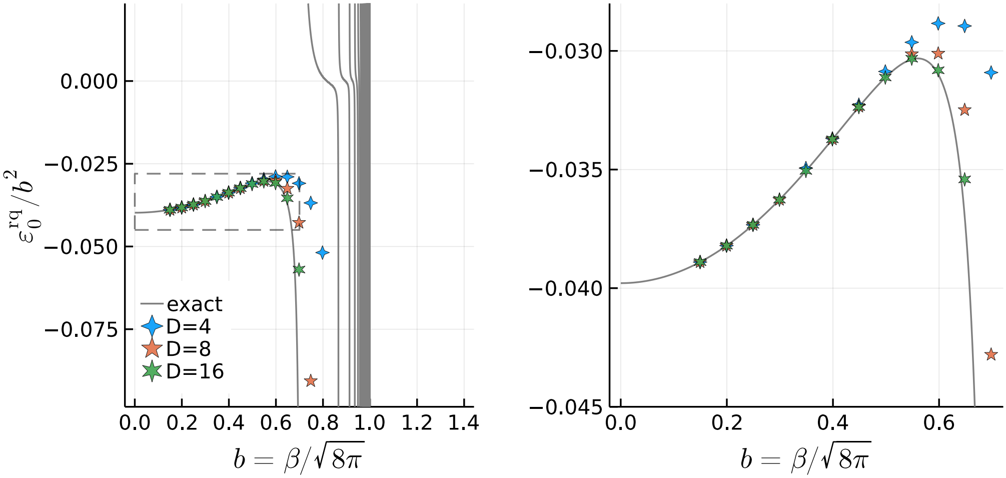

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Numerically refines Coleman's argument from $b = 1$ to $b = 1/\sqrt{2} + \epsilon(D)$

Getting serious with $\cosh(\beta\phi)$

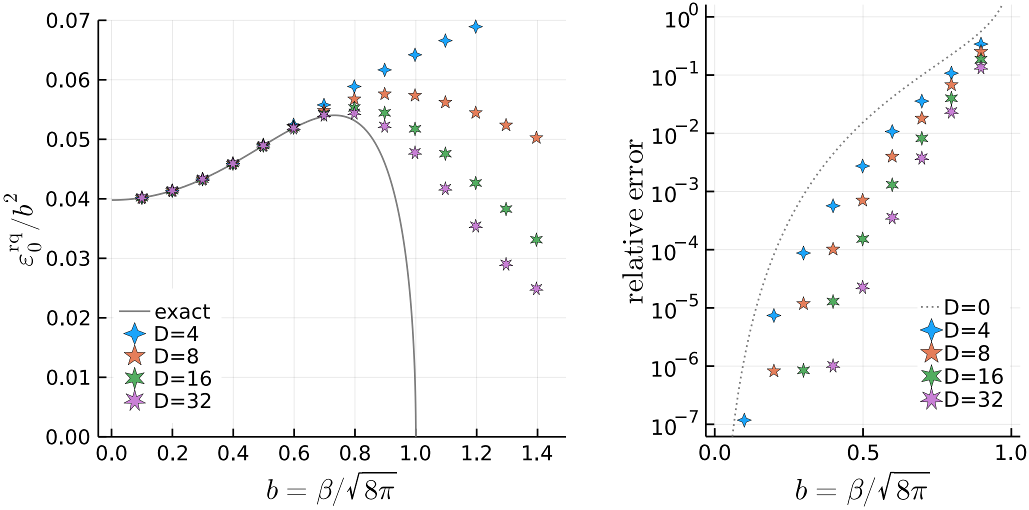
The Sinh-Gordon model

Renormalized Hamiltonian of $\cos(\beta\phi)$ theory

$$H = \int \mathrm{d}x \, \frac{:\pi^2:_m}{2} + \frac{:(\nabla\phi)^2:_m}{2} + \frac{m^2}{\beta^2} :\cosh(\beta\phi):_m$$

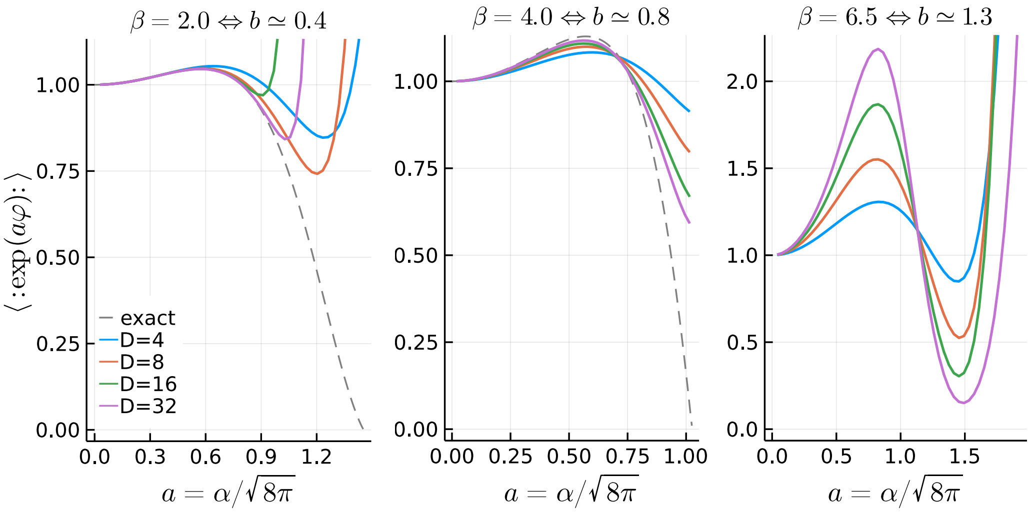
1. Constructed rigorously by Fröhlich and Park for $b = \beta/\sqrt{8\pi} < 1/\sqrt{2}$
2. No value of b at which the potential is obviously ill-defined
3. Analytical results for all $b \geqslant 0$, likely valid only for $b \leqslant 1$ (or even just $b \leqslant 1/\sqrt{2}$?)
4. Conjectured to be massless for $b \geqslant 1$ by KLM and BLC
5. One can **try** RCMPS for all $b \geqslant 0$

Results: (rescaled) energy density

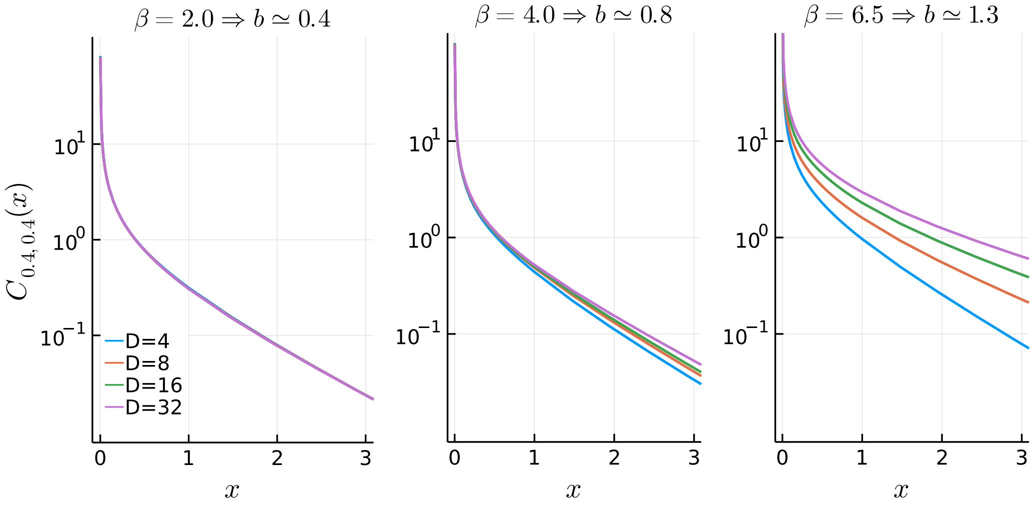


Results: vertex operators $\langle : e^{a\varphi} : \rangle$

Known exactly from FLZZ formula up to $a = (b + b^{-1})/2$ (Seiberg bound)



Results: 2-point func $\langle :e^{a\varphi(x)}: :e^{a\varphi(0)}:\rangle - \langle :e^{a\varphi(x)}:\rangle\langle :e^{a\varphi(0)}:\rangle$



Discussion and open problems

Understanding expressiveness of RCMPS

Standard Entanglement Entropy

Defined for “standard” locality

$$\rho_{\geqslant 0} = \int \prod_{x \leqslant 0} \mathrm{d}\phi(x) \langle \phi | \Psi \rangle \langle \Psi | \phi \rangle$$

Gives $S_1 = -\mathrm{tr}\left(\rho_{\geqslant 0} \log \rho_{\geqslant 0}\right) \propto \log(\Lambda)$

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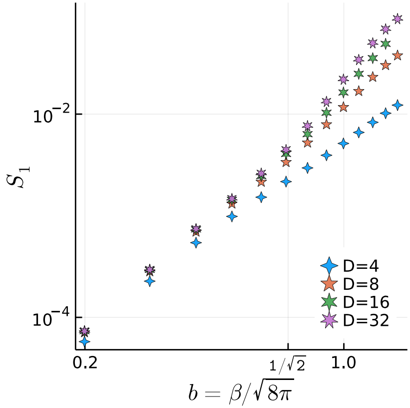
Exotic Entanglement Entropy

Defined for RCMPS notion of locality

trace over $a^\dagger(x_1) \cdots a^\dagger(x_n)|0\rangle_m$ for $x_k \leqslant 0$

Gives $S_1 = O(1)$ (numerically)



EEE is finite at least for

$b \leqslant 1/\sqrt{2}$

Sinh-Gordon theory: what do we know?

Still uncertainty, following KLM, BLC, and the present study...

Personnally think

1. 99% chance: Hamiltonian H has no self-duality $b \to b^{-1}$
2. 80% chance: Any reasonable definition of the model is massless for $b \geqslant 1$
3. 70% chance: Energy formula correct for $b \in [0, 1]$, and $e_0 = 0$ for $b \geqslant 1$.
4. 50% chance: FLZZ formula correct for *all* $a \geqslant (b + b^{-1})/2$
5. 50% chance: The model makes sense, without renormalization, for $b \geqslant 1$
6. 50% confidence: UV fixed point does not change for $b \geqslant 1$

Open problems: rigorously construct the model for $b \geqslant 1/\sqrt{2}$ / Find if it has an entanglement phase transition

Todo-list for continuous tensor networks

In $1+1$ dimensions

- Solve Fermion / Gauge theories
- Go into the $b \geqslant 1/\sqrt{2}$ of Sine-Gordon
- Do general CFT perturbations
- Compute more observables (masses, spectra, c-function...)

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Remaining objectives do higher dimensions!

	non-relativistic	relativistic	critical
$d=1$ space	Verstraete-Cirac 2010	AT 2021	
$d \geqslant 2$ space	AT-Cirac 2019		

no idea	heuristics	clear definition	fast algorithm

Summary

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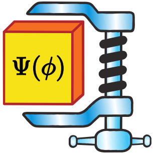

1. Ansatz for $1+1$ relativistic QFT
2. No cutoff, UV or IR, extensive, computable
3. Efficient (cost poly D, error 1/superpoly D)
4. Rigorous (variational)
5. Works well for ϕ^4_2, Sine-Gordon, and Sinh-Gordon at $b \leqslant 1/\sqrt{2}$