ERC Starting Grant 2021
Part B2

Section a. State of the art and objectives

a1. Introduction

Context – the continuous many-body problem

Quantum systems with many degrees of freedom interacting non-trivially make the fabric of our world. Yet it is currently very difficult predict their behaviors when interactions are not small, *i.e.* when degrees of freedom get strongly correlated, an unfortunately ubiquitous situation. Intuitively, the root of the problem lies in the exponential explosion of the size of the Hilbert space as the number of degrees of freedom or particles increases. The many-body problem gets even more thorny when there is no lattice to discretize space, as the Hilbert space is infinite even before the exponential increase with the number of particles.

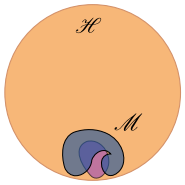
This difficulty of the continuous many-body problem restricts our capacity to understand the fundamental constituents of nuclear matter from first principles, with quantum chromodynamics (QCD). At an other extreme of energy, it limits our understanding of strongly coupled topological phases of matter like the fractional quantum Hall effect (FQHE) [1]. In both cases, it is believed the laws of physics are known to a sufficient level of accuracy, but it is our inability to solve them that restricts the predictive power of theoretical physics.

For such problems, perturbation theory cannot be used (interactions are strong) and thus the most natural method is to discretize space or space-time and then use the Monte Carlo method. In the context of quantum field theory, lattice gauge theory has given us unprecedented access to non-perturbative phenomena in QCD [2–4]. It is a powerful but unfortunately costly method, that converges slowly and that suffers from the sign problem in some instances (*e.g.* with large chemical potential). Hence, even with exascale computing, many observables of interest will remain out of reach of lattice Monte Carlo [5]. It is thus crucial to develop new analytical and numerical methods to tackle the continuum many-body problem.

There are two main deterministic alternatives one should consider: the variational method, and non-perturbative renormalization group approaches. The first is the main focus of this proposal while the second will be explored as a natural byproduct in the very last work package (WP 3.2).

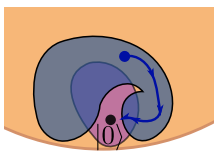
The variational method

The Hilbert space of many-body systems is too large to be manageable directly and even infinite in the continuum. If we could *compress* states in this Hilbert space down to a small (finite) number of parameters, then things would get easier. In what follows I will restrict myself to the ground state search problem, but the method extends to the low lying spectrum [6, 7], thermal states [8, 9], and even dynamics [10]. For this problem, the variational method proceeds in essentially two steps:



1. *Compression* – based on physical insights, *guess* a finite dimensional submanifold $\mathcal{M} \subset \mathscr{H}$ of the Hilbert space (aka an ansatz wavefunction) that should contain (or be close to) the states of interest,

2. *Minimization* – minimize the energy (expectation value of the Hamiltonian H) over this submanifold, *e.g.* through gradient descent, to find an approximation of the ground state



$$|0\rangle \simeq \operatorname*{argmin}_{|\psi\rangle\in\mathcal{M}} \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}\,. \tag{1}$$

By increasing the dimension of $\mathcal{M}$, and provided the physical guess is good, one can refine the approximation arbitrarily, and get as close as needed to the correct ground state.

There are 3 desirable properties for the submanifold $\mathcal{M}$ that were noted by Feynman in 1987 [11] and that I reformulate in a modern way here:

1. *Extensivity* – as the size L the system grows, or the number of particle N grows, the size of the Hilbert space $\mathscr{H}$ grows exponentially. The submanifold should however be parameterized extensively, that it its dimension should grow at most linearly as the system grows (or at worse polynomially)

2. *Computability* – for a state $|\psi\rangle \in \mathcal{M}$, there should be an efficient method to compute expectation values $\langle\psi|\mathcal{O}|\psi\rangle$ that does not require explicitly summing over the (exponentially or infinitely many) states in the Hilbert space basis. This is required if only to compute the energy and minimize it.

3. *Short distance regularity (for relativistic QFT)* – the minimization problem should be well defined, and minimizing a (possibly divergent) energy density should not yield a runaway behavior where the approximation at physically revelant lengthscales degrades through the optimization.

The last requirement is the most technical and subtle. It is related to the singular short distance behavior of relativistic quantum field theories and is important only in this specific context.

In fact, many popular ansatz wave-functions do not verify these requirements. Hamiltonian truncation [12–16] uses as submanifold a *vector space* which makes the minimization step trivial (it is a linear eigenvalue problem) but breaks extensivity (the cost of the method is $\propto \mathrm{e}^{LE_T}$ where E_T is a truncation energy). The recently introduced neural network wavefunctions [17, 18] are sparsely parameterized, but break the second requirement of computability: their expectation values have to be evaluated by carrying the sum over states approximately with Monte Carlo. Both methods can still be efficient for a wide class of problems, *e.g.* because numerical prefactors are favorable despite prohibitive asymptotics or because importance sampling is efficient in many situations of interest.

However, *if it is possible for the problem considered*, it is clearly better to fit the 3 requirements above: the method will then typically be better behaved and reliable (because of computability), not requiring a finite size cutoff (because of extensivity) and not requiring a short distance cutoff (because of regularity).

Tensor network states on the lattice

Tensor network states provide a submanifold fitting precisely the first two requirements of Feynman on the lattice (where the third is irrelevant). Generic states are not compressible, in the same way that a picture with random pixels cannot be efficiently compressed with JPEG. There is a compression possible because the states we target are *highly unusual*, they are the low energy states of local Hamiltonians. Such states are much more locally entangled than typical states in the Hilbert space, and verify the *area law*. More precisely, for a low energy state of a local Hamiltonian, the entanglement entropy $S_{\mathcal{R}}$ of a regular connected subregion
$mathcalR$ of space with the rest grows proportionally to the border of $\mathcal{R}$, *i.e.* $S_{\mathcal{R}} \propto |\partial\mathcal{R}|$ (with additional logarithmic factors in the gapless case) [19–21]. Typical states in the Hilbert space (*e.g.* sampled with uniform measure over rays) verify instead a volume law, *i.e.* $S_{\mathscr{D}} \propto |\mathscr{D}|$ [22, 23]. This gives the heuristic picture in Fig. 1 below.

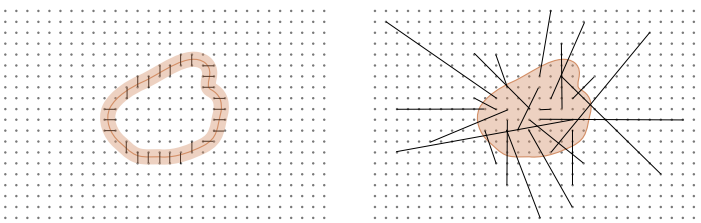


Figure 1: Heuristic representation of entanglement between a closed subregion of the lattice and the rest, for a low energy state of a local gapped Hamiltonian (left) and a typical random state (right).

Tensor network states leverage this crucial insight from quantum information theory to target directly this corner of the Hilbert space verifying the area law. Targetting this corner with a *polynomial reduction in entanglement* enables a *local* parameterization of states, and thus an *exponential compression* of the number of parameters.

Concretely, tensor network states are obtained from a product of an extensive number of low rank tensors $A^j_{k_1 k_2 \dots k_r}$ with the indices k contracted along the links of a given network [24]. The contracted indices k are called bond indices, and the number of values they can take is called the *bond dimension* D. The larger the bond dimension, the larger the submanifold dimension and number of

adjustable parameters. The remaining free index j is called physical, and corresponds the local Hilbert space basis states.

One can illustrate how Feynman's requirements are met with a simple example on the line. Consider a chain of N qudits (local Hilbert space $\mathbb{C}^d$). A generic quantum state for this chain is

$$|\psi\rangle = \sum_{j_1,j_2,\cdots,i_N=0}^{d} c_{j_1 j_2 \cdots j_N} |j_1, j_2, \cdots, j_N\rangle\,, \tag{2}$$

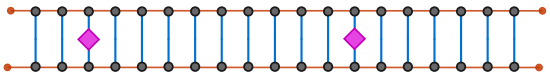
and thus requires d^N complex coefficients $c_{j_1 j_2 \cdots j_N}$ to be specified (or equivalently a rank N tensor c). An example of tensor network ansatz that is adapted if we are looking for low energy states of a gapped Hamiltonian is the matrix product states (MPS) [25]. They are obtained from the contraction of a rank 3 tensor A^j_{kl}

$$|A\rangle = \sum_{j_1,j_2,\cdots,i_N=0}^{d} (L|A^{j_1} A^{j_2} \cdots A^{j_N}|R)\,|j_1, j_2, \cdots, j_N\rangle \tag{3}$$

where $(L|$ and $|R)$ are two arbitrary vectors $\in \mathbb{C}^D$. Here, the contraction of the lower indices of A is given by the matrix product. At this stage it is more convenient to use a graphical notation where $A^{j}_{\mathbf{kl}} =$  and connected lines correspond to a contraction of indices, *i.e.* k  $l = \sum \delta_{kl}$. With this notation, the wave-function c, and by abuse of notation the state itself, read

$$|A\rangle \longleftrightarrow c_{j_1 j_2 \cdots j_N} =$$  , (4)

where the orange dots at the endpoints correspond to the final scalar product with the boundary vectors. The state parameterization is local. In the translation invariant case, total number of parameters is independent of the system size $d \times D^2 \ll d^N$ (and grows only linearly $d \times D^2 \times N$ in the general case). **This solves Feynman's first requirement of extensive parameterization.** One can easily show that MPS verify the area law, and thus target the right corner of the Hilbert space. Expectation values are also efficiently computable, for example the correlation function of two local observables $\mathcal{O}$ reads in graphical notation

$$\langle A|\mathcal{O}(j_a)\mathcal{O}(j_b)|A\rangle =$$  , (5)

where the purple squares are the tensor notation for local $\mathcal{O}$ (matrices on physical indices, or rank 2 tensors). This ladder tensor network is contractible with a number of operations that does not grow exponentially with N, as it requires only the (iterated) multiplication of two different maps

$$\Phi =$$  and $\Phi_{\mathcal{O}} =$  . (6)

This solves Feynman's second requirement of efficient computability. One can further show that MPS give good approximations for ground states of local gapped Hamiltonians. More precisely, minimizing the energy density over all MPS of bond dimension D, one gets an approximation to the ground state with an error for local observables that decreases superpolynomially [26, 27], *i.e.* that is $o(D^{-\alpha})$ for *all* $\alpha > 0$.

In addition with MPS, the most important tensor network states that have been proposed are the projected entangled pair states (PEPS) [28], adapted to 2 space dimensions and more, and the multiscale entanglement renormalization ansatz (MERA) [29–31], adapted to critical (gapless) systems in 1 space dimension (see Fig. 2 below).

PEPS also provide a local (extensive) parameterization of states, Feynman's first requirement is thus naturally satisfied. They also verify the area law. However computing expectation values is not as straightforward as for MPS with formula (5). The two dimensional equivalents of the maps Φ and $\Phi_{\mathcal{O}}$ in (6) have a size that still grows exponentially with N. Consequently, there exists no general purpose exact method of contraction [32]. We instead have efficient controllable approximation routines that converge fast for most problems [33]. Thus Feynman's second requirement is also satisfied for this ansatz, albeit in a weaker form.

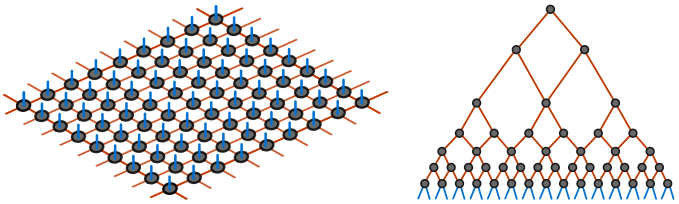


Figure 2: PEPS on the left, adapted to gapped systems in 2 space dimensions (or more), and MERA on the right, adapted to gapless systems in 1 space dimensions. As before, the bond indices contracted over are shown in orange, while physical indices are shown in blue.

a2. State of the art in tensor networks for the variational method

On the lattice

Tensor networks have allowed tremendous formal progress in theoretical physics, allowing *e.g.* the succint classification of symmetry protected [34–37] and topological [38, 39] phases of matter. Here instead I would like to focus specifically on their use in the variational approach.

In 1 space dimension, the popular density matrix renormalization group (DMRG) [40, 41], put forward in the early nineties, can be reinterpreted as a specific minimization over a submanifold of MPS, which retrospectively explains its impressive success. It is fair to say that for 1 dimensional lattice systems with a gapped Hamiltonian, the problem of finding low energy states, the corresponding spectrum, and thermal states, is essentially *solved* by MPS (with DMRG or other optimization methods). Many papers now quote MPS results as *numerically exact*, since the error is well understood theoretically, arbitrarily refinable, and below machine precision for many observables of interest. This is the ideal situation that one should now aim to extend to the continuum.

In 2 space dimensions or more (but still on the lattice), progress with PEPS is more recent. Minimizing the energy over the state parameters is more difficult [33] but still doable efficiently in practice. Further, the field has recently benefited from the introduction of automatic differentiation techniques [42] to compute descent directions, simplifying many computations. A wide range of models have been studied, including the highly non-trivial Fermi Hubbard model in $\mathbf{d} = 2$ space dimensions [43, 44]. Apart from the quest for ever increasing precision, a major open problem is the difficulty to write the wavefunctions of (gapped) chiral topological phases as PEPS [45]. As we will see, there is a natural way to do it in the continuum (WP1.2), which provides yet another motivation to leave the lattice.

For critical systems on the lattice, state of the art results have been obtained with 3 classes of techniques. The first is to use standard tensor networks like MPS or PEPS, that are *not a priori* adapted to gapless phases, but to carry a finite entanglement scaling [46–48] (similar to a finite size scaling) to extract predictions. The second road was to use tensor network implementations of renormalization group techniques [49, 50]. Taking this route to the continuum is the subject of WP 3.2. Finally, the last option has been to use a state more difficult to manage but directly adapted to critical systems [29, 31], the MERA, with impressive numerical results as well. It is this latter way we aim to extend to the continuum in WP3.1.

Recent progress in the continuum

Numerical continuum limit – The most straightforward way to use tensor network states in a continuum context is simply to discretize the model under study, solve it variationally with the standard lattice toolbox, take finer and finer lattices, and numerically extrapolate to the continuum. This way, a wide variety of continuous models have been studied with tensor networks. To cite only a few, relativistic quantum field theories have been studied with MPS (self-interacting scalar field [51], Schwinger model [52–54], Gross-Neveu model [55]) and impressive results have been obtained for interacting non-relativistic fermions with PEPS [56]. Moving away from purely variational approaches into renormalization group methods, but still using tensor networks, the self interacting scalar field has been studied to exquisite precision (an effort I contributed to [57]), beating the Monte Carlo method [58, 59] in low dimensions. Recently, it was even pushed to higher dimensions [60], although getting close to the continuum limit proved harder.

In all these works and as I observed myself, the computational cost substantially increases as the lattice spacing is reduced, at least for relativistic theories. As one gets closer to the continuum limit, the local effective physical dimension and bond dimensions explode for a fixed required precision (one needs to take huge elementary tensors) [57]. The approach can still be exceptionally precise in practice thanks to extrapolations, but there is a limit to how fine a lattice one can take.

One may wonder if this limitation is inevitable. In the continuum, the Hilbert space becomes continuously infinite, and thus one may conjecture that no compression, no matter how efficient, can bring down this infinity down to a finite number of parameters. The difficulty of fulfilling Feynman's third requirement in relativistic QFT –where arbitrarily short distances seem to swallow all the variational parameters– would seem to further support this conjecture. The surprising conclusion of recent work done directly in the continuum is that this conjecture would be wrong! One can work in the continuum limit directly, without the number of parameters blowing up.

Continuous matrix product states – The pioneering result showing that the continuum was reachable directly was obtained in 2010 by Verstraete and Cirac [61]. They took the continuum limit of MPS themselves and obtained a finite dimensional submanifold of states for non-relativistic QFT, the continuous matrix product states (CMPS). For bosons on the line $[0, L]$ with associated local creation operator $\hat{\psi}(x)$ such that $[\hat{\psi}(x), \hat{\psi}^\dagger(y)] = \delta(x-y)$, a CMPS is a state parameterized by two $D \times D$ complex matrices Q and R and that reads:

$$|Q, R\rangle = \mathrm{tr}\left[\mathcal{P}\exp\left\{\int_0^L \mathrm{d}x\, Q \otimes \mathbb{1} + R \otimes \hat{\psi}(x)\right\}\right]|0\rangle_\psi\,, \tag{7}$$

where $\mathcal{P}\exp$ is the path-ordered exponential, $|0\rangle_\psi$ is the Fock vacuum associated to the $\hat{\psi}$ operators, *i.e.* $\forall x,\ \hat{\psi}(x)|0\rangle_\psi = 0$, and the trace is taken over the matrix space. The types of Hamiltonians this ansatz can be applied to are non-relativistic, *i.e.* typically of the form of the Lieb-Liniger Hamiltonian

$$H_{\mathrm{LL}} = \int_0^L \partial_x\hat{\psi}^\dagger\partial_x\hat{\psi} - \mu\hat{\psi}^\dagger\hat{\psi} + g\hat{\psi}^\dagger\hat{\psi}^\dagger\hat{\psi}\hat{\psi} \tag{8}$$

which happens to be integrable [62], but non-integrable deformations work just as well [63]. For such problems, expectation values are computable exactly [64] and optimizing over CMPS gives extremely precise results [65], directly in the continuum. The needed optimization algorithms have been refined in the past decade and reach good performance in a wide variety of situations [66].

This ansatz however comes with two limitations

1. $\mathbf{d} = 1$ *only* – the generalization to space dimensions $\mathbf{d} \geq 2$ is not straightforward
2. *Non-relativistic* – the ansatz does not work for relativistic QFT (expectation values are divergent)

These two restrictions have confined CMPS to a rather small niche. Despite some attempts [67], I think it is fair to say that progress in pushing CMPS beyond its niche has been limited in the past decade (until recently). My main recent contribution in this domain, which makes the present proposal viable, has been to lift these two limitations.

Continuous tensor network states – **In 2018, I put forward continuous tensor network states (CTNS) with Ignacio Cirac** [68]. The main insight was to accept that a radically different construction was needed for space dimensions $\mathbf{d} \geq 2$. Indeed, in the continuum limit in $\mathbf{d} = 1$, coarse graining keeps the bond dimension fixed but increases the local physical dimension (see Fig. 3 below). This is why we can describe quantum fields with using only a discrete (matrix) auxiliary space. In $\mathbf{d} \geq 2$, the bond-dimension increases as well upon coarse graining. Hence to get a meaningful continuum limit, the bond degrees of freedom have to become fields as well.

Exploiting this insight, and taking the continuum limit carefully, we came up with the following definition. In $\mathbf{d}$ space dimensions, for a non-relativistic QFT living in $\Omega \subset \mathbb{R}^{\mathbf{d}}$, a CTNS is a state parameterized by two functions V and α (and a boundary functional B) and given by

$$|V, B, \alpha\rangle = \int \mathcal{D}\phi\, B(\phi_{\partial\Omega}) \exp\left\{-\int_{\Omega\subset\mathbb{R}^{\mathbf{d}}} \mathrm{d}^{\mathbf{d}}x\, \frac{[\nabla\cdot\phi(x)]^2}{2} + V[\phi(x)] - \alpha[\phi(x)]\,\hat{\psi}(x)\right\}|0\rangle_\psi\,. \tag{9}$$

The field ϕ here is an auxiliary scalar field (or a vector of scalar fields), and the functional integral means that it is given by a Euclidean QFT in $\mathbf{d}$ space-time dimensions. Note that this is one dimensions *less* than the non-relativistic theory living in $\mathbf{d}+1$ space-time that we target. Of course, it is not sufficient to solve the auxiliary theory for ϕ to solve the physical one, one also needs to optimize the parameters

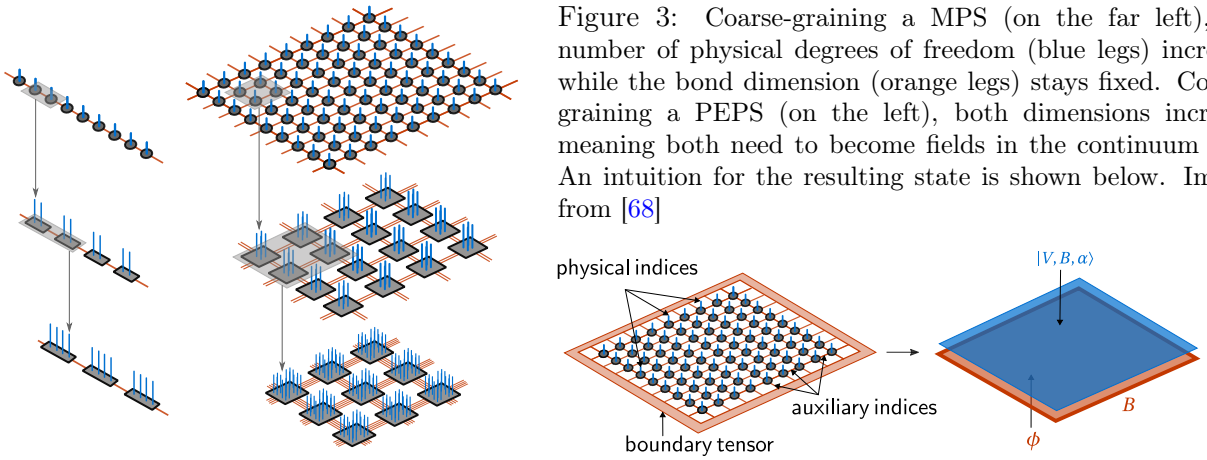


Figure 3: Coarse-graining a MPS (on the far left), the number of physical degrees of freedom (blue legs) increases while the bond dimension (orange legs) stays fixed. Coarse-graining a PEPS (on the left), both dimensions increase, meaning both need to become fields in the continuum limt. An intuition for the resulting state is shown below. Images from [68]

variationally. We get something similar to what happens with tensor networks on the lattice and *trade a space dimension for a variational optimization.* Crucially, even if bond degrees of freedom ϕ are now fields (in one dimension less), they can be parameterized with a finite number of parameters, *e.g.* by expanding the functions V (analog of Q) and α (analog of R) as polynomials in ϕ.

Computing expectations values on the state $|V, B, \alpha\rangle$ with generic parameters cannot be done exactly. Just like with discrete PEPS, one needs approximate contraction algorithms that do not exist yet in the continuum. However, for a subset of parameters, namely $V[\phi]$ at most quadratic in ϕ and $\alpha[\phi]$ at most linear in ϕ, the state $|V, B, \alpha\rangle$ is Gaussian, and thus we have accessed to exact formulas. For such states, we could carry a full numerical optimization on some toy models with my student Teresa Karanikolaou [69]. We observed excellent (and provable) convergence, and exactly the right short distance behavior. This showed that CTNS provide the right *compression*. The general *minimization* algorithm for non-Gaussian states –which would allow to exploit this compression power for generic problems– does not exist yet. Its development corresponds to the work packages WP1.1 and WP1.3 of the present proposal.

Going relativistic – Whilst CMPS address Feynman's first two requirements for non-relavistic theories, they fail at the third in the relativistic context. This was noted already in 2010 by Haegeman et al. [70], and the solution, or rather patch, was to regulate the high energy behavior by adding a non-relativistic cutoff in the Hamiltonian. While functional, this trick partially defeated the purpose of going to the continuum in the first place, since there was still some extrapolation to do on the final results to lift the UV (short distance) regulator.

In early 2021, I came up with a solution of this problem, and finally addressed Feynman's last requirement with a relativistic extension of CMPS (RCMPS) [71, 72]. The idea was to change the field operator $\hat{\psi}^\dagger(x)$ appearing in the state definition into the slightly non-local $a^\dagger(x) = \int \mathrm{d}p\, e^{ipx} a_p^\dagger$ where $a_p^\dagger$ is the operator creating *free* particles over the "bare" ground state of the free part H_0 of the Hamiltonian H (the part at most quadratic in the fields). Why this is exactly the right choice is not completely obvious but understood and discussed in [72]. What matters is that with this choice, computations get much more involved *while remaining feasible*, and all the divergences disappear. In a nutshell, the state fits the short distance behavior *exactly* out of the box, preventing the kind of runaway behavior Feynman feared. Using these states, I could carry the full optimization and compute observables for the self-interacting scalar field aka ϕ_2^4 theory. Numerical results then confirmed the fast convergence of the error to zero as the bond dimension D increased (cost polynomial in D, error most likely superpolynomial in D). This was the first time a relativistic QFT was solved *non-perturbatively* without any cutoff UV (short distance) or IR (long distance). This result supports the whole WP2 of this proposal, that would otherwise seem like science fiction.

Ansatz for critical systems – The continuous matrix product state and its recent higher dimensional and relativistic generalizations are adapted to gapped systems (massive QFT). There has been attempts at constructing a continuous equivalent of the multiscale entanglement renormalization ansatz, a CMERA, that would be adapted to gapless systems (critical QFT). An influential proposal was put forward by Haegeman et al. in 2011 [73]. It received a tremendous interest in high energy

physics because of its link with holography [74]. However, its practical interest has been more limited. Indeed, explicit computations are feasible only for Gaussian states and the extensions are typically only perturbative [75]. Further, the method is not adapted to relativistic QFT, that is, while the long distance behavior of these states can be made critical, the short distance behavior is regular as in a non-relativistic QFT. One can safely say that these states have probably not reached their final form yet, and the purpose of WP3.1 is to do better and construct a new relativistic CMERA that turns this approach into something concretely useful.

Summary: The extension of tensor network states to the continuum, was carried in 2010 with CMPS in the non-relativistic $\mathbf{d} = 1$ space dimension case. It was the first compression from an infinite to a finite finite number of parameters, but it could be applied only to a niche of systems. **Very recent developments obtained by the PI** show that one need not stay in that niche: $\mathbf{d} \geq 2$ is feasible (although no associated minimization algorithm exists yet) and, even more surprisingly perhaps, relativistic QFT can be dealt with as well, at least for $\mathbf{d} = 1$ space dimension. Finally, ansatz for critical systems in the continuum are not mature yet in any dimension but a third breakthrough is possible.

a3. Objectives

The situation – to go beyond the first promises, a scale up is needed

After the very recent progress I described, solving the many-body problem in the continuum, for its most interesting physically relevant instances, no longer seems delusional. However, it does remain a formidable challenge to go beyond the low-dimensional toy models. For example, solving ϕ_2^4 with relativistic CMPS was already rather non-trivial, but we need much more efforts (analytic and algorithmic) to get closer to real-world theories. This effort is something I can no longer do alone, or only at the cost of substantially delaying progress. It is the right time to scale up the effort, and push continuous tensor networks into the mainstream.

Long term goal – tackle the relevant hard problems

The long term goal is to build a variational method to numerically solve most physically relevant strongly correlated many-body quantum systems in the continuum. This method, based on continuous tensor network states, will then yield a host of analytic insights as well into these systems, as it did on the lattice previously.

These physically relevant problems which are currently out of reach are numerous, but they approximately fall into 3 categories: i) non-relativistic, ii) relativistic, and iii) critical. In the first category, typical problems are strongly correlated topological phases of matter, like the fractional quantum Hall effect, and interacting quantum gases. In the second one, the archetypal example is quantum chromodynamics (QCD), the theory of the strong force. In the last one, the interesting problems are related to $\mathbf{d} = 2$ critical quantum systems (or $\mathbf{d} = 3$ classical statistical systems), for which the best method currently available is the conformal bootstrap [76]. Finding a robust variational method for even a single of these problems would be a major breakthrough.

In the long run, the hope is that continuous tensor network states can allow substantial progress on all these physically relevant problems, either by directly allowing their solution or by being part of a solution.

Practical goals – making CTNS competitive, mainstream, or promising

The three categories of hard problems I presented were shown in a decreasing order of maturity for continuous tensor network states (CTNS). The non-relativistic setting is now better understood, with the $\mathbf{d} = 1$ case done in 2010, and the $\mathbf{d} \geq 2$ ansatz I introduced with Cirac in 2018 [68]. What needs to be done is to construct a generic routine to (approximately) compute observables, develop an energy minimization algorithm, and construct an extension of the class of states to deal with chiral topological phases. This means the most relevant and important problems are within reach of this project. My

goal is to obtain substantial quantitative progress on these problems, and make continuous tensor network states competitive, numerically and analytically, with respect to alternatives.

In the relativistic setting, the $\mathbf{d} = 1$ case is more recent (2021) and there is a lot to develop in this context already. No $\mathbf{d} \geq 2$ ansatz exists yet, and dealing with gauge theories adds further subtleties that are yet to be addressed. As a result, my main objective is to lay the groundwork that will subsequently make relevant QFT like QCD solvable. To this end, I will attack each hurdle independently first in different low dimensional toy models. The project should provide sufficient progress to make CTNS mainstream in relativistic QFT, and confirm the path towards QCD is viable. Further, results in $\mathbf{d} = 1$ relativistic QFT are already particularly important in themselves, for their use in quantum information and quantum technologies (quantum wires, quantum transmission lines with delay, etc.). Finally, for critical problems, there is not even a fully convincing ansatz in $\mathbf{d} = 1$. The kind of advance done in 2010 and 2021 for non-relativistic and relativistic problems is still required. But I am confident it can be found, or at least that it is worth trying much harder. The practical objective here will be more modest: showing a proof of principle that such systems can be tackled variationally directly in the continuum.

Philosophy – do *not* require more structure than is needed

On the path towards relevant problems, we will solve toy models, that contain some but not all of their features. This can be a sterile exercise if the ideas developed for toy models cannot in principle scale up, or exploit peculiar ad-hoc structures present only in the models.

My philosophy will be to keep the methods as generic as possible, both in their analytic and numerical use. I will not try to exploit peculiar structures like integrability that are not generic. This distinguishes the program I propose from popular research lines that try to understand the QFT via exact solutions [77]. There is one feature shared by *all* the systems we are interested in, that is needed for compression, and that we will exploit: a local entanglement structure.

Summary: After recent proofs of principle obtained by the PI, it is now clear that attacking strongly correlated quantum systems directly in the continuum with tensor networks is viable. However, it remains a formidable task (requiring intense effort) to **do it for physically relevant problems**: this is our **long term objective**. The more practical goals reachable in 5 years differ depending on the application: we should aim for the state of the art on non-relativistic problems, reaching the mainstream on toy relativistic problems, and obtaining new proofs of principle for critical problems. In all instances, the philosophy is to build general robust methods, that do not rely on integrability.

Section b. Methodology

b1. WP1 – Non-relativistic field theory

The non-relativistic context is the most mature, long term objectives are already within reach. It is divided in 3 work packages (WP): two more towards computational goals (1.1 and 1.3), and one more formal (1.2) but that provides new applications to the other two.

WP1.1 – Efficient contraction methods and algorithms in 2 space dimensions

The main hurdle to apply continuous tensor networks in $\mathbf{d} \geq 2$ is the lack of an approximate routine to compute local expectation values, which are needed (if only) for optimization. For a CTNS $|V, B, \alpha\rangle$, there is no general algorithm to estimate functions like $\langle V, B, \alpha|\hat{\psi}^\dagger(x)\,\hat{\psi}(y)|V, B, \alpha\rangle$. The most direct way to compute them, and the one I plan to follow, is to use the *dimensional reduction power of tensor network states* [78]. More precisely, computing the correlation functions of a CTNS in $\mathbf{d}$ is reduced to finding the stationary state of a transfer matrix acting on two copies of a $\mathbf{d} - 1$ space dimension *relativistic* QFT Hilbert space (see Fig. 4 below).

This method was foreshadowed in [68]. The fixed point of the transfer matrix can be found variationally, which is now a problem we know how to solve! In the end, an efficient relativistic CMPS

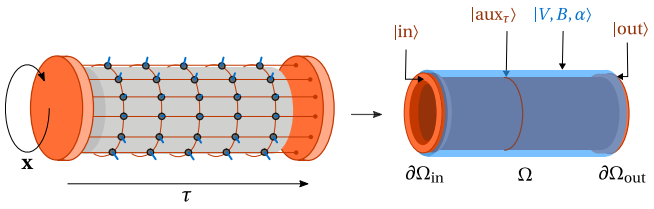


Figure 4: Illustration of the dimensional reduction allowed by discrete and continuous tensor networks in a $\mathbf{d} = 2$ cylinder. One of the space direction of the physical theory becomes an imaginary time direction of the auxiliary (bond) relativistic field theory. In correlation functions, one gets two copies of the auxiliary theory, coupled by the connection of physical indices. Image from [68]

optimization algorithm (for $\mathbf{d} = 1$ relativistic problems) provides a powerful contraction method in $\mathbf{d} = 2$ space dimension (for non-relativistic problems). This connects this WP1.1 and the WP2.1 related to relativistic field theories.

These two WP are particularly adapted to a PhD student **[PhD 1]** right at the beginning of the project, because no new ansatz is a priori needed and one starts from a rather well defined basis. Under my close supervision, the student will explore the relativistic CMPS and their optimization for this particular problem and keep an open mind at alternatives computation routes not detailed here (inspired from other discrete techniques, *e.g.* the corner transfer matrix renormalization group). *The main result will be a clean, open-source, and easily usable algorithm to compute CTNS expectation values.*

WP1.2 – Fractional Quantum Hall (FQH) wave-functions with (chiral) continuous tensor network states

This WP is a more theoretical intermezzo aimed at developing a natural application of CTNS by *generalizing* the ansatz provided in [68]. An important open problem is to construct variational wave-functions for chiral topological phases, a task which seems impossible with PEPS tensor networks on the lattice. The idea is to use the same type of ansatz as in (9) but replacing ϕ (a scalar field) by φ a chiral massless field. This gives a state that typically has the form:

$$|\Psi\rangle = \langle 0|\mathcal{R} \exp\left[\int \mathrm{d}^2 z\, \mathcal{V}_\beta(z) \otimes \hat{\psi}(z)\right] |0\rangle_{\mathrm{CFT}} \ \ |0\rangle_\psi \tag{10}$$

where $|0\rangle_{\mathrm{CFT}}$ is the chiral free boson CFT vacuum, $\mathcal{V}_\beta =: \mathrm{e}^{i\beta\varphi(z)} :$ is a vertex operator, $\varphi(z)$ is the chiral free boson, and $\mathcal{R}$ is the CFT radial ordering operator. This is the analog of the standard CTNS definition in (9) where $\int \mathcal{D}\phi \cdot \exp(\int (\nabla\phi)^2/2) \to \langle 0|\mathcal{R} \cdot |0\rangle_{\mathrm{CFT}}$, $V \to 0$, and $\alpha[\phi(z)] \to \mathcal{V}_\beta(z)$. Up to a screening term not shown for clarity, this state $|\Psi\rangle$ in (10) is a continuous tensor network representation of the Laughlin state for $\beta = \sqrt{n}$, $n \in \mathbb{N}^*$, a prototypical topological wave-function built from CFT correlation functions. *CTNS contain and generalize a wide range of previously used ansatz wave-functions.* The construction I just presented *is not published yet*, but a similar one for $p_x \pm ip_y$ super conductors was introduced by Dubail, Read, and Rezayi [79] (before CTNS even existed). More generally, one should be able to write *all* the ansatz wave functions for the quantum Hall effect introduced so far as (chiral) continuous tensor network states.

Making this connection precise, and exploiting its analytic and numerical consequences will be done in this WP. The first natural analytic consequence is a connection of so far disconnected ways of understanding topological phases from the field theory [80] and tensor network [39] point of views. Providing a clear dictionary will be a major advance in the field. The numerical consequence may become even more important. Ansatz wave-functions like the Laughlin state (and other states built from CFT correlators) are exact [39], but their correlation functions are not efficiently computable *a priori*. In fact the best method is to use MPS on a thin cylinder [81], which scales in this context $\propto \exp\sqrt{N}$ where N is the number of particles in the state. This is better than the naive $\propto \exp(N)$ scaling of exact diagonalization, but still superpolynomial. Here, the CTNS mapping combined with the results of WP1.1 should give an algorithm with a cost constant in N, *i.e.* usable directly in the continuum and thermodynamic limits. *The WP will establish the general dictionary, and show on a proof of principle that it allows efficient computations (at least) for the Laughlin state.*

This WP1.2 is a natural task for a second PhD student **[PhD 2]** more interested in formal developments, and who would (ideally) start 1 or 2 years after the first. This WP requires excellent knowledge of $2d$ CFT, and is thus particularly synergistic with WP3.1.

WP1.3 – Full variational algorithm for 2 space dimensions

In this WP, I will build on WP1.1 to obtain a full energy minimization algorithm for $\mathbf{d} = 2$ problems using CTNS. To this end, one needs to be able to compute expectation values *fast*, so that they can be used as a routine to be called many times during the minimization. This will likely require an improvement from the algorithm obtained in WP1.1 to target speed (*e.g.* through parallelization). To make the subsequent minimization efficient, it is faster to use *geometric* optimization methods (see *e.g.* [7]), which require computing a gradient and a metric. The latter can also be expressed as CTNS expectation values, which shows the central role of a fast algorithm to evaluate them.

At first, for minimization, we could use standard geometric methods used in the field of tensor networks, like the time dependent variational principle (TDVP) [82]. But note that several researchers at the host institution *Centre d'automatique et systèmes* at *Mines Paris* happen to be mathematicians experts on geometric optimization problems, and thus could provide crucial insights. In particular, it is likely possible to combine TDVP with quasi Newton methods like BFGS, which would allow to faster optimization. I will work on establishing this collaboration between physicists and mathematicians for the benefit of this work package.

This WP is ambitious, but it ultimately allows to go beyond pure ansatz wavefunctions (without optimizations) and toy Gaussian problems, to solve a wide range of physically relevant systems. It is natural for this WP to be carried by [**PhD 1**] at the end of their PhD, building upon the knowledge acquired with WP1.1. *The main result of this WP should be an efficient open source algorithm that does better than state of the art lattice techniques (e.g. [56]) on* $\mathbf{d} = 2$ *interacting gases of Fermions or Bosons.*

Summary of WP1 - non-relativistic field theory: In this WP, we will develop methods to make the established compression power of continuous tensor network states (CTNS) finally computationally useful in $\mathbf{d} \geq 2$ space dimensions. Namely, we will develop an algorithm exploiting the dimensional reduction to compute expectation values (WP1.1) and then find ground states (and other observables) through minimization (WP1.3). A crucial intermezzo (WP1.2) will allow to enlarge the class of CTNS to include chiral topological phases which cannot be dealt with in the discrete.

b2. WP2 – Relativistic field theory

In the relativistic context, we are already at the stage of the proof of principle, and the objective is now to make the technique mainstream, on the road towards the long term goals (including *e.g.* QCD). To this end the WP is divided in 3: the first for a thorough exploration of low dimensional QFT which are easier to treat but still largely unexplored, the second to adapt continuous tensor networks to gauge fields, which are crucial in fundamental physics, and the last one to develop a new ansatz in higher dimension and definitely escape toy models.

WP2.1 – The $1+1$ dimensional case beyond the first example

Despite the very recent progress I mentioned [71], relativistic QFT in $\mathbf{d} = 1$ space dimension remained largely unexplored since only a single non-trivial example (ϕ_2^4 theory) was so far studied. A more encyclopedic understanding of what could happen would be useful. This WP will provide a more thorough study of the $\mathbf{d} = 1$ case by solving a broader class of theories, and by developing the tools needed to explore more observables (including *e.g.* spectra). Apart from the intrinsic interest of such theories, this study is important because i) it is synergistic with WP1.1 as an efficient and sufficiently *general* algorithm allows to compute correlation functions for CTNS in more dimensions ii) because $\mathbf{d} = 1$ theories have usually been studied with other methods (like Hamiltonian truncation), and solving them with a new method with typically better asymptotic properties is a good way to draw the attention of the community to the approach.

Concretely, in this WP, we will develop a general procedure to solve a class of $\mathbf{d} = 1$ *relativistic QFT that is as large as possible. Then we will apply this framework to solve a wide range of example, in a methodical manner.* Currently, bosonic field theories with polynomial potentials (*e.g.* ϕ^6, ϕ^8, $\lambda\phi^3 + g\phi^4$,...) can be studied with the existing ansatz (although this remains to be done). For more

general models of interest with potentials like $\mu\sin(b\phi)$ or $\mu\sinh(b\phi)$ –which are integrable and thus where there are many alternative benchmarks (and surprisingly open questions as well [83])– one still needs to find an efficient way to compute the energy expectation values. For Fermionic theories, it is possible that the ansatz works already, but it is likely that one needs to fix extra regularity conditions as for non-relativistic theories. Finally, an interesting perspective is to study field theories that are *not* the relevant perturbation of a free theory (as is common in high-energy physics), but instead a relevant perturbation of a generic CFT (like a minimal model).

It is possible that we cannot manage all theories, but the method will be as general as possible, and we will have a clear theoretical understanding of why some (if any) models need a more subtle approach.

This open-ended WP will likely be carried in two steps, with progress expected through the whole duration of the project. It is natural for the first PhD student [**PhD 1**] to study simple Fermionic theories and bosonic theories with more generic potentials together with the development of an algorithm for WP1.1. The study of more exotic theories like relevant perturbations of generic CFT, is more naturally carried after, by someone with a fine understanding of CFT. It is thus natural as a second project for the second PhD student [**PhD 2**].

WP2.2 – Incorporating gauge field degrees of freedom

On the way to treating difficult relativistic QFT like quantum chromodynamics (QCD), one needs to deal with gauge theories. Concretely, the state one constructs need to be annhilated by a gauge operator which fixes the gauge condition. In the past years, it was shown how such a non-trivial condition can be implemented into tensor network states on the lattice (with the first proposals, and recent proof of principle computations in $\mathbf{d} = 1$ and $\mathbf{d} = 2$). *In this WP, we will generalize the idea to the continuum, by constructing gauge invariant states for field theories, and then apply it to toy models in low dimensions.* A first application will be the Schwinger model, *without* using the trick to eliminate gauge degrees of freedom (this would be against the *philosophy* I put forward of using methods that are generic).

The strategy will be to follow closely the various methods to define gauge invariant states on the lattice [84–87], and then take the scaling limit in the same way as for continuous tensor networks. Knowing the standard definition of gauge theories in the continuum also gives a lot of insight about what the continuum limit should look like, and we will use it to find the right scaling.

If the postdoc [**postdoc 1**] allocated to this project WP2 happens to be an expert of gauge theories in the Hamiltonian formalism, this will be a natural project for them. Some experience is important as this WP consists in *inventing* a new ansatz, and not only in computing better with existing ones. If [**postdoc 1**] has profile fitting better the following sub WP, I will establish a collaboration with existing groups working on this problem in the discrete, *e.g.* at the University of Ghent (with Jutho Haegeman), at the University of Barcelona (with Luca Tagliacozzo) or at the Max Planck Institute of quantum optics.

WP2.3 – Developing an ansatz for relativistic QFT in $2+1$ dimensions

In this WP, the objective is to put forward a new class of states, directly in the continuum, adapted to relativistic QFT in $\mathbf{d} \geq 2$. In a nutshell, this means putting together the progress from [68] (that brought CMPS from $\mathbf{d} = 1$ to $\mathbf{d} \geq 2$) and [71, 72] (that made the $\mathbf{d} = 1$ case relativistic). This is however not a simple fusion, and this is one of the most ambitious WP.

The main difficulty lies in the fact that in $\mathbf{d} = 2$ space dimensions ($\mathbf{d}+1$ spacetime), relativistic QFT have (in general) more complex divergences than in $\mathbf{d} = 1$. Consequently, merely defining the Hilbert space on which the QFT Hamiltonian acts is a non-trivial task (it is no longer the free Fock space). I believe the most promising way to attack this problem is to follow the tracks of very recent work done in $\mathbf{d} = 2$ with renormalized Hamiltonian truncation. This method is very different from tensor networks, but it relies on the Hamiltonian formalism and thus faces the same issues. The way researchers in this field model the problem rigorously in the Hamiltonian formalism, –dealing with a very unusual way to formulate renormalization–, is likely to be an excellent source of inspiration.

This is a WP for which I would like to allocate a postdoc [**postdoc 1**] if they are not already on the gauge theory WP. If the profile of [**postdoc 1**] fits better the previous gauge theory WP, then

I would rely primarily on an external collaboration with Hamiltonian truncation experts (*e.g.* Joan Elias-Miro at ICTP).

Summary of WP2 - relativistic field theory: We will go beyond the first example and thoroughly explore QFT in $1+1$ space-time (WP2.1). The output will be a clean and general method, instantiated in an open source algorithm, to carry computations in the most general QFT possible. We will then include gauge degrees of freedom into the description (WP2.2), and finally put forward a new CTNS ansatz for higher dimensions (WP2.3). These are the 3 main bricks that will be needed to apply CTNS in high energy physics and QCD in particular, which is the long term goal.

b2. WP3 – Critical systems

For critical systems, there is no convincing proof of principle using tensor networks directly in the continuum and this is what we aim to obtain in this WP. In the first part (WP3.1) we will stay with the variational method and simply change of submanifold to find a class of states adapted to gapless models out of the box. Since this part is the most early stage, it is also the more open in terms of method. Thus, in the second part (WP3.2), we will depart from the variational method to explore tensor renormalization approaches, and develop their promising continuum analogs.

WP3.1 – Construct an ansatz for critical QFT in $1+1$ spacetime dimensions

In this WP, we will construct a new class of states adapted to the ground states of relativistic QFT at their critical point in $\mathbf{d}=1$ space dimension. This ansatz will allow computations with controlled approximations, in the general case and not jut at Gaussian fixed points. This will distinguish this class of states from the current continuous multiscale entanglement renormalization ansatz (CMERA) which i) works for non-relativistic theories (trivial short distance behavior) ii) does not allow non-perturbative computations out of the Gaussian case.

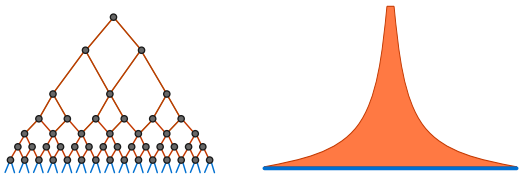


Figure 5: Right: a MERA tensor network. Left: the type of continuous equivalent we are looking for. Image from [68]

Note, importantly, that a critical relativistic QFT (like the self interacting scalar field at its critical point) is *not* a CFT. A critical relativistic QFT is a CFT 1 (UV) at short distance, another CFT 2 (IR) at long distances, and something complicated and non-trivial at intermediate length-scales. Crucially, a relativistic QFT at criticality still has a scale (equivalent to the lattice scale for a non-relativistic critical lattice model).

The ansatz we will put forward will be adapted to the short distance behavior of the theory by being defined in the Hilbert space of CFT 1 (UV). One option (which may or may not turn out the be the right one), will be to consider a construction similar to the CMERA but written in terms of CFT operators:

$$|\Psi\rangle = \mathcal{P}\exp\left[i\int_0^{s_{\max}} \mathrm{d}s\, D + \mathcal{O}(s)\right]|0\rangle_{\mathrm{UV}} \tag{11}$$

where s is a scale, D is the (relativistic) dilation operator in the UV CFT, and $\mathcal{O}(s)$ is an operator (disentangler) made from a local integral of fields in the UV CFT. Such a state $|\Psi\rangle$ will live in the UV CFT Hilbert space, but, upon optimization, its correlation functions at large distances will get close to that of another CFT. This second IR CFT will have a lower central charge c, and this may give an alternative proof or intuition for Cardy's c-theorem.

This WP requires an excellent knowledge of CFT in 1+1 dimensions, is synergistic with WP1.2, and will thus be a natural second project for **[PhD 2]**. The WP also requires creativity since the objective is to introduce a new class of states that achieves the right compression, and thus some maturity. A postdoc **[postdoc 2]** allocated to the whole WP, with a transverse vision on critical systems will thus

be extremely useful. *At the end of the project, we should have a new class of states and associated optimization routine, tested on the simplest QFT like ϕ_2^4 at criticality, and giving crucial insights into critical phenomena in the continuum.*

WP3.2 – Tensor renormalization in the continuum

Most of this project is dedicated to tensor networks as a compression ansatz for the variational method. Another popular use is to compute partition functions in lattice statistical model through a powerful instantiation of the renormalization group (RG). In this WP, we will construct new continuum versions of this tensor renormalization approach, which will then allow to understand critical systems as fixed points.

On the lattice, the idea of tensor based renormalization techniques comes from noting that the partition function of a lattice model is exactly a tensor network. One may then locally coarse grain the elementary tensors to obtain a flow in the space of tensors. This is the analog of the Midgal-Kadanoff RG [88], where one flows coupling constants instead of a tensor. The advantage of the tensor formulation is that there is an easy way to define an "optimal" local truncation, typically through a singular value decomposition (SVD) of the elementary tensor (as in the simple tensor renormalization group, or TRG). With these methods, one knows how much local error one makes locally. Ultimately, and with the recent refinements [50], this makes the tensor network implementations of the RG far more precise, and for many problems the most accurate method available for numerics [57]!

Such a construction does not exist yet in the continuum for generic QFT, although an attempt has been made [89]. There is most likely no direct continuum limit *in scale* of the lattice methods, because a local coarse graining changes the scale by a fixed discrete factor (typically 2), whereas we want a continuous scale evolution. But my objective is to find methods that are *inspired* from the optimal truncation techniques of the lattice. A possible strategy is to start from the functional renormalization group [90], which has recently shown accurate results for scalar field theories [91], but find a better (and dynamically defined) truncation scheme to replace the derivative expansion (mimicking the role played by the SVD).

This is likely the most open ended WP of this proposal, pushing beyond variational methods. It could lead to either a completely new accurate non-perturbative renormalization method, or merely a refinement of the functional RG. As for the previous WP, this work requires a transverse understanding of renormalization and critical systems which will typically be the skills required for the postdoc [**postdoc 2**] who will work on it with me.

Summary of WP3 - critical systems: In WP3.1, we will show a proof of principle that critical systems can be dealt with directly in the continuum with an appropriate CTNS ansatz. It will be built by putting together the tools of conformal field theory and earlier tensor network attempts (CMERA). In WP3.2, we will use ideas from tensor networks on the lattice to find new non-perturbative formulations of the renormalization group in the continuum, with a controllable precision.

b4. Perspectives

The starting point of this project is the insight that it is possible to efficiently *compress* the infinite dimensional Hilbert spaces of continuous many-body quantum systems. A priori, this compression can be used to *solve* many difficult and physicall relevant instances of strongly coupled quantum systems (which verify approximately the area law). This project will allow to solve some of them, and build the elementary bricks needed to solve the others. Along the way, it will also bring the variational method to many illuminating toy models of theoretical physics. This will tightly connect quantum information science, condensed matter theory, and high energy physics together.

Our project will also have direct applications for *implementations* of quantum technologies and the concrete problems explored (*e.g.* quantum transmission lines with delay, well modeled by $\mathbf{d} = 1$ QFT) in the QUANTIC group, a joint research effort between ENS Paris, Inria, and the host institution.

Conversely, the our new set of methods will bring important theoretical progress in connection with the development quantum computing. Many of the problems we mentioned, including relativistic

QFT, are often mentioned as natural *applications* of quantum computers. What our methods will show is that classical methods can work as well for a wide variety of low energy observables (where states verify the area law). This will show where pure quantum gains are possible: precisely where the tensor compression no longer works.

b5. Summary and connection between the work packages

The different work packages and the PhD students and postdocs allocated to them are summarized in Fig. 6 below.

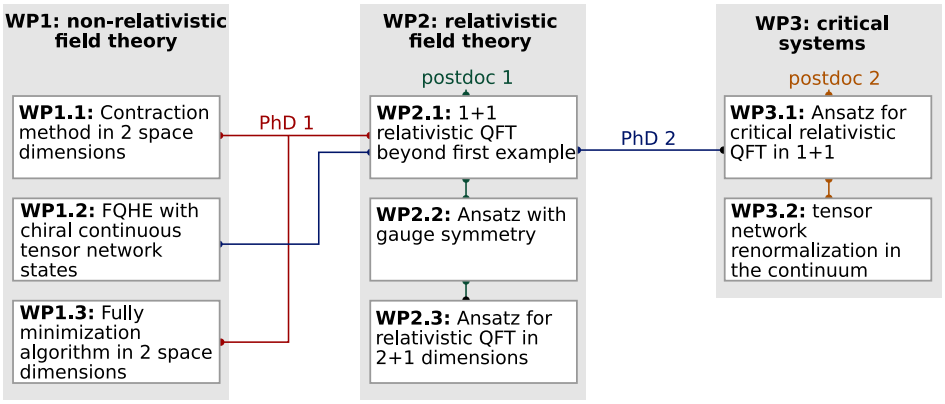


Figure 6: Summary of the work packages and their connections. There is a horizontal connection via common methods, made concrete by the tasks given to PhD students, and a vertical connection by objectives, made concrete by the tasks given to postdocs.

The estimated start and end dates of the work packages are shown in Fig. 7 below.

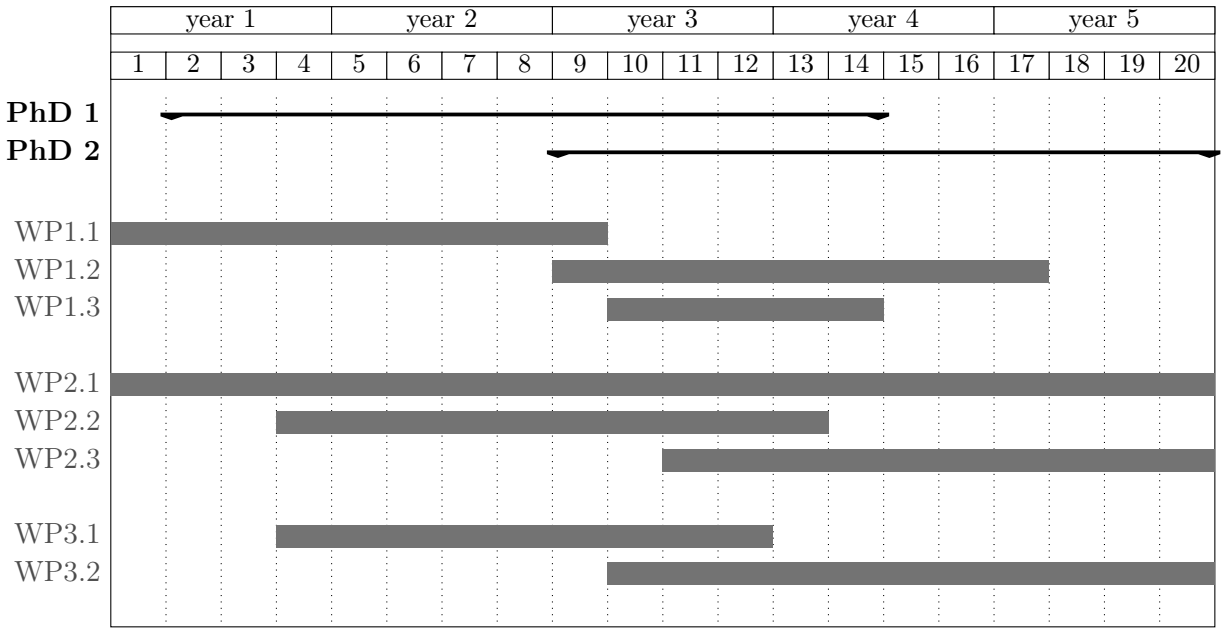


Figure 7: Gantt chart of the project. The start and finish dates of WP2.2, WP2.3 and WP3 are more flexible and determined by the exact recruiting time of the 2 corresponding postdocs.

References

[1] M. E. Cage, K. Klitzing, A. Chang, F. Duncan, M. Haldane, R. Laughlin, A. Pruisken, and D. Thouless, *The quantum Hall effect* (Springer Science & Business Media, 2012).

[2] M. Creutz, L. Jacobs, and C. Rebbi, Physics Reports **95**, 201 (1983).

[3] FLAG Working Group, S. Aoki, Y. Aoki, C. Bernard, T. Blum, G. Colangelo, M. Della Morte, S. Dürr, A. X. El-Khadra, H. Fukaya, R. Horsley, A. Jüttner, T. Kaneko, J. Laiho, L. Lellouch, H. Leutwyler, V. Lubicz, E. Lunghi, S. Necco, T. Onogi, C. Pena, C. T. Sachrajda, S. R. Sharpe, S. Simula, R. Sommer, R. S. Van de Water, A. Vladikas, U. Wenger, and H. Wittig, The European Physical Journal C **74**, 9 (2014).

[4] J. B. Kogut, Rev. Mod. Phys. **55**, 775 (1983).

[5] B. Joó, C. Jung, N. H. Christ, W. Detmold, R. G. Edwards, M. Savage, and P. Shanahan, EPJA **55**, 199 (2019).

[6] L. Vanderstraeten, J. Haegeman, and F. Verstraete, Phys. Rev. B **99**, 165121 (2019).

[7] L. Vanderstraeten, J. Haegeman, and F. Verstraete, SciPost Phys. Lect. Notes , 7 (2019).

[8] F. Verstraete, J. J. García-Ripoll, and J. I. Cirac, Phys. Rev. Lett. **93**, 207204 (2004).

[9] A. Kshetrimayum, M. Rizzi, J. Eisert, and R. Orús, Phys. Rev. Lett. **122**, 070502 (2019).

[10] L. Hackl, T. Guaita, T. Shi, J. Haegeman, E. Demler, and J. I. Cirac, SciPost Phys. **9**, 48 (2020).

[11] R. P. Feynman, Difficulties in applying the variational principle to quantum field theories, in *Variational Calculations in Quantum Field Theory* (World Scientific Publishing, Singapore, 1987) pp. 28–40.

[12] A. J. A. James, R. M. Konik, P. Lecheminant, N. J. Robinson, and A. M. Tsvelik, Reports on Progress in Physics **81**, 046002 (2018).

[13] J. Elias-Miró, M. Montull, and M. Riembau, Journal of High Energy Physics **2016**, 1–34 (2016).

[14] J. Elias-Miró, S. Rychkov, and L. G. Vitale, Phys. Rev. D **96**, 065024 (2017).

[15] J. Elias-Miró, S. Rychkov, and L. G. Vitale, JHEP **2017** (10), 213.

[16] S. Rychkov and L. G. Vitale, Phys. Rev. D **91**, 085011 (2015).

[17] G. Carleo and M. Troyer, Science **355**, 602 (2017).

[18] K. Choo, A. Mezzacapo, and G. Carleo, Nature communications **11**, 1 (2020).

[19] M. Srednicki, Phys. Rev. Lett. **71**, 666 (1993).

[20] M. M. Wolf, F. Verstraete, M. B. Hastings, and J. I. Cirac, Phys. Rev. Lett. **100**, 070502 (2008).

[21] J. Eisert, M. Cramer, and M. B. Plenio, Rev. Mod. Phys. **82**, 277 (2010).

[22] D. N. Page, Phys. Rev. Lett. **71**, 1291 (1993).

[23] O. C. O. Dahlsten, C. Lupo, S. Mancini, and A. Serafini, Journal of Physics A: Mathematical and Theoretical **47**, 363001 (2014).

[24] J. C. Bridgeman and C. T. Chubb, J. Phys. A: Math. Theor. **50**, 223001 (2017).

[25] M. Fannes, B. Nachtergaele, and R. F. Werner, Commun. Math. Phys. **144**, 443 (1992).

[26] F. Verstraete and J. I. Cirac, Phys. Rev. B **73**, 094423 (2006).

[27] Y. Huang, Computing energy density in one dimension (2015), arXiv:1505.00772 [cond-mat.str-el] .

[28] F. Verstraete and J. I. Cirac, Renormalization algorithms for quantum-many body systems in two and higher dimensions (2004), arXiv:cond-mat/0407066 [cond-mat.str-el] .

[29] G. Vidal, Phys. Rev. Lett. **99**, 220405 (2007).

[30] S. Montangero, M. Rizzi, V. Giovannetti, and R. Fazio, Phys. Rev. B **80**, 113103 (2009).

[31] G. Evenbly and G. Vidal, in *Strongly Correlated Systems. Numerical Methods*, edited by A. Avella and F. Mancini (Springer-Verlag, Berlin DE, 2013) Chap. 4, pp. 99–130.

[32] J. Haferkamp, D. Hangleiter, J. Eisert, and M. Gluza, Phys. Rev. Research **2**, 013010 (2020).

[33] S.-J. Ran, E. Tirrito, C. Peng, X. Chen, L. Tagliacozzo, G. Su, and M. Lewenstein, Two-dimensional tensor networks and contraction algorithms, in *Tensor Network Contractions: Methods and Applications to Quantum Many-Body Systems* (Springer International Publishing, Cham, 2020) pp. 63–86.

[34] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Phys. Rev. B **81**, 064439 (2010).

[35] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **83**, 035107 (2011).

[36] N. Schuch, D. Pérez-García, and I. Cirac, Phys. Rev. B **84**, 165139 (2011).

[37] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **87**, 155114 (2013).

[38] N. Schuch, J. I. Cirac, and D. Pérez-García, Ann. Phys. **325**, 2153 (2010).

[39] N. Bultinck, M. Mariën, D. Williamson, M. Şahinoğlu, J. Haegeman, and F. Verstraete, Annals of Physics **378**, 183 (2017).

[40] S. R. White, Phys. Rev. Lett. **69**, 2863 (1992).

[41] S. R. White, Phys. Rev. B **48**, 10345 (1993).

[42] H.-J. Liao, J.-G. Liu, L. Wang, and T. Xiang, Phys. Rev. X **9**, 031041 (2019).

[43] P. Corboz, Phys. Rev. B **93**, 045116 (2016).

[44] B.-X. Zheng, C.-M. Chung, P. Corboz, G. Ehlers, M.-P. Qin, R. M. Noack, H. Shi, S. R. White, S. Zhang, and G. K.-L. Chan, Science **358**, 1155 (2017), https://science.sciencemag.org/content/358/6367/1155.full.pdf .

[45] J. Dubail and N. Read, Phys. Rev. B **92**, 205307 (2015).

[46] F. Pollmann, S. Mukerjee, A. M. Turner, and J. E. Moore, Phys. Rev. Lett. **102**, 255701 (2009).

[47] V. Stojevic, J. Haegeman, I. P. McCulloch, L. Tagliacozzo, and F. Verstraete, Phys. Rev. B **91**, 035120 (2015).

[48] B. Vanhecke, J. Haegeman, K. Van Acoleyen, L. Vanderstraeten, and F. Verstraete, Phys. Rev. Lett. **123**, 250604 (2019).

[49] G. Evenbly and G. Vidal, Phys. Rev. Lett. **115**, 180405 (2015).

[50] M. Hauru, C. Delcamp, and S. Mizera, Phys. Rev. B **97**, 045111 (2018).

[51] A. Milsted, J. Haegeman, and T. J. Osborne, Phys. Rev. D **88**, 085030 (2013).

[52] M. C. Bañuls, K. Cichy, J. I. Cirac, and K. Jansen, JHEP **2013** (11), 158.

[53] M. C. Bañuls, K. Cichy, K. Jansen, and H. Saito, Phys. Rev. D **93**, 094512 (2016).

[54] M. C. Bañuls, K. Cichy, J. I. Cirac, K. Jansen, and S. Kühn, Phys. Rev. Lett. **118**, 071601 (2017).

[55] G. Roose, N. Bultinck, L. Vanderstraeten, F. Verstraete, K. V. Acoleyen, and J. Haegeman, Lattice regularisation and entanglement structure of the gross-neveu model (2021), arXiv:2010.03441 [hep-lat] .

[56] R. Haghshenas, Z.-H. Cui, and G. K.-L. Chan, Numerical continuum tensor networks in two dimensions (2020), arXiv:2008.10566 [cond-mat.str-el] .

[57] C. Delcamp and A. Tilloy, Phys. Rev. Research **2**, 033278 (2020).

[58] W. Loinaz and R. S. Willey, Phys. Rev. D **58**, 076003 (1998).

[59] S. Bronzin, B. De Palma, and M. Guagnelli, Phys. Rev. D **99**, 034508 (2019).

[60] S. Akiyama, Y. Kuramashi, and Y. Yoshimura, Phase transition of four-dimensional lattice ϕ^4 theory with tensor renormalization group (2021), arXiv:2101.06953 [hep-lat] .

[61] F. Verstraete and J. I. Cirac, Phys. Rev. Lett. **104**, 190405 (2010).

[62] E. H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963).

[63] J. Rincón, M. Ganahl, and G. Vidal, Phys. Rev. B **92**, 115107 (2015).

[64] J. Haegeman, J. I. Cirac, T. J. Osborne, and F. Verstraete, Phys. Rev. B **88**, 085118 (2013).

[65] M. Ganahl, J. Rincón, and G. Vidal, Phys. Rev. Lett. **118**, 220402 (2017).

[66] B. Tuybens, J. D. Nardis, J. Haegeman, and F. Verstraete, Variational optimization of continuous matrix product states (2020), arXiv:2006.01801 [quant-ph] .

[67] D.Jennings, C. Brockt, J. Haegeman, T. J. Osborne, and F. Verstraete, New J. Phys. **17**, 063039 (2015).

[68] A. Tilloy and J. I. Cirac, Phys. Rev. X **9**, 021040 (2019).

[69] T. D. Karanikolaou, P. Emonts, and A. Tilloy, Gaussian continuous tensor network states for simple bosonic field theories (2020), arXiv:2006.13143 [cond-mat.str-el] .

[70] J. Haegeman, J. I. Cirac, T. J. Osborne, H. Verschelde, and F. Verstraete, Phys. Rev. Lett. **105**, 251601 (2010).

[71] A. Tilloy, Variational method in relativistic quantum field theory without cutoff (2021), arXiv:2102.07733 [quant-ph] .

[72] A. Tilloy, Relativistic continuous matrix product states for quantum fields without cutoff (2021), arXiv:2102.07741 [quant-ph] .

[73] J. Haegeman, T. J. Osborne, H. Verschelde, and F. Verstraete, Phys. Rev. Lett. **110**, 100402 (2013).

[74] M. Nozaki, S. Ryu, and T. Takayanagi, Journal of High Energy Physics **2012**, 10.1007/jhep10(2012)193 (2012).

[75] J. S. Cotler, M. Reza Mohammadi Mozaffar, A. Mollabashi, and A. Naseh, Phys. Rev. D **99**, 085005 (2019).

[76] D. Poland, S. Rychkov, and A. Vichi, Rev. Mod. Phys. **91**, 015002 (2019).

[77] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, L. Freyhult, N. Gromov, R. A. Janik, V. Kazakov, T. Klose, and et al., Letters in Mathematical Physics **99**, 3–32 (2011).

[78] T. J. Osborne, J. Eisert, and F. Verstraete, Phys. Rev. Lett. **105**, 260401 (2010).

[79] J. Dubail, N. Read, and E. H. Rezayi, Phys. Rev. B **86**, 245310 (2012).

[80] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. **80**, 1083 (2008).

[81] M. P. Zaletel and R. S. K. Mong, Phys. Rev. B **86**, 245305 (2012).

[82] J. Haegeman, J. I. Cirac, T. J. Osborne, I. Pižorn, H. Verschelde, and F. Verstraete, Phys. Rev. Lett. **107**, 070601 (2011).

[83] R. Konik, M. Lájer, and G. Mussardo, JHEP **2021** (1).

[84] E. Rico, T. Pichler, M. Dalmonte, P. Zoller, and S. Montangero, Phys. Rev. Lett. **112**, 201601 (2014).

[85] L. Tagliacozzo, A. Celi, and M. Lewenstein, Phys. Rev. X **4**, 041024 (2014).

[86] J. Haegeman, K. Van Acoleyen, N. Schuch, J. I. Cirac, and F. Verstraete, Phys. Rev. X **5**, 011024 (2015).

[87] E. Zohar, M. Burrello, T. B. Wahl, and J. I. Cirac, Ann. Phys. **363**, 385 (2015).

[88] L. P. Kadanoff, Annals of Physics **100**, 359 (1976).

[89] Q. Hu, A. Franco-Rubio, and G. Vidal, Continuous tensor network renormalization for quantum fields (2018), arXiv:1809.05176 [hep-th] .

[90] C. Wetterich, Physics Letters B **301**, 90 (1993).

[91] I. Balog, H. Chaté, B. Delamotte, M. Marohnić, and N. Wschebor, Phys. Rev. Lett. **123**, 240604 (2019).