Bootstrapping the stationary state of bosonic open quantum systems

with Gustave Robichon

Is it just cute?

<u>Problem</u> :

$$\frac{d\rho_t}{dt} = \mathcal{L}(\rho_t)$$

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_k \mathcal{D}[c_k](\rho)$$

$$D[c](\rho) = c\rho c^\dagger - \frac{1}{2}\{c^\dagger c, \rho\}$$

$\mathcal{H}$ is bosonic Hilbert space

$$\mathcal{L}(\rho) = -i\omega[a^\dagger a, \rho] + k_1 \mathcal{D}[a](\rho) + k_2 \mathcal{D}[a^2 - \alpha^2 I](\rho)$$

<u>Question</u> : compute ρ_∞ stationary state

$$\mathcal{L}(\rho_\infty) = 0$$

$$\boxed{\langle O \rangle_\infty := \mathrm{tr}[O \rho_\infty]}$$

<u>Standard approach</u> : $\mathcal{H}_{N_c} = \mathrm{span}\{a^{\dagger k}|0\rangle, k \leq N_c\}$

$\hookrightarrow$ • $\tilde{\rho}_\infty = \lim_{t \to +\infty} e^{t\tilde{\mathcal{L}}} \tilde{\rho}_{in}$ → RK, Krylov, Rauchan 2

• $\tilde{\mathcal{L}} \tilde{\rho}_\infty = 0$ $\longrightarrow$ Arnoldi method

$\hookrightarrow$ cost $\propto N_c^2$

Some issues :

1. cost explodes when $\langle n \rangle$ large

2. error bounds are not rigorous

3. $\| \rho_a - \tilde{\rho}_\infty \|$ small

$\hookrightarrow$ $\mathrm{tr}[\underbrace{\theta}_{\downarrow\ \mathrm{poly}(a, a^\dagger)} \rho_\infty]$ well approximate!

use semi-definite relaxations

II) Sum of squares

$$\underbrace{P(x_1, \ldots, x_n)}_{\text{degree } 2d} = \sum_k \underbrace{Q_k(x_1, \ldots, x_n)}_{\text{degree } d \text{ max}}^2 + \lambda$$

$$\Longrightarrow \quad P \geqslant \lambda \quad \Rightarrow \quad \min P > \lambda$$

- Apply this idea to the Lindblad case

normal ordered monomial

$$\mathcal{S}_D = \{\underbrace{I}_{0}, \underbrace{a, a^\dagger}_{1}, \underbrace{a^{\dagger 2}, a^2, a^\dagger a}_{\text{degree } 2}, \underbrace{a^{\dagger 3}, \dots}_{\text{degree } 3} \underbrace{\quad}_{D}\}$$

$$= \{ O_i \} \qquad \mathcal{S}_D \subset \mathcal{S}_{D+1}$$

Imagine we SOS certificate:

$$O = \alpha I + \sum_i \beta_i \mathcal{L}^\dagger(O_i) + \sum_{ij} \gamma_{ji} O_i^\dagger O_j \qquad \gamma \geq 0$$

this equality proves $\mathrm{tr}[O \rho_\infty] > \alpha$

Proof:

$$\mathrm{tr}[O\rho_\infty] = \underbrace{\mathrm{tr}[\alpha\rho_\infty]}_{\alpha} + \sum_i \beta_i \underbrace{\mathrm{tr}[\mathcal{L}^\dagger(O_i)\rho_\infty]}_{\substack{= \mathrm{tr}[O_i \mathcal{L}(\rho_\infty)] \\ = 0}}$$

$$+ \underbrace{\sum_{ij} \underset{\gamma \geq 0}{\gamma_{ji}} \underbrace{\mathrm{tr}[O_i^\dagger O_j \rho_\infty]}_{M_{ij}}}_{\mathrm{tr}[\gamma M] \geq 0} \qquad M \geq 0$$

$$\geq \alpha_{\mathrm{lower}}$$

↳ find similar writing for $-O = \alpha_{\mathrm{upper}} I + \ldots$

$$\boxed{\alpha_{\mathrm{lower}} \leq \mathrm{tr}[O\rho_\infty] \leq \alpha_{\mathrm{upper}}}$$

$\longrightarrow$ • $\quad \mathcal{O} = \sum_k B_k \, \mathcal{O}_k$ $\qquad$ e.g. $\mathcal{O} = a^\dagger a$

$\Longrightarrow B_k = \delta_{k6}$

$\longrightarrow$ • use commutation relations $[a, a^\dagger] = 1$

$$\mathcal{O}_i^\dagger \mathcal{O}_j = \sum_k \underbrace{A_{ij}^k}_{\text{constants, don't depend on } \mathcal{L}} \mathcal{O}_k$$

normal ordered

$\longrightarrow$ • $\quad \mathcal{L}^\dagger(\mathcal{O}_i) = \sum_{k=1}^{f(i)} L_i^k \, \mathcal{O}_k$

$\in \mathcal{J}_D$

$\mathcal{J}_{D+|\mathcal{L}|}$

encode dynamics

here assuming the $\mathcal{L}$ contains finite power of $a, a^\dagger$

$$\hookrightarrow \mathcal{O} = \alpha J + \sum_{i \in \mathcal{L}_D} \beta_i \mathcal{L}^{t}(\mathcal{O}_i) + \sum_{\substack{i,j \\ \in D}} \gamma_{ji} \mathcal{O}^{t}_i \mathcal{O}_j$$

$$\sum_k B_k \mathcal{O}_k = \alpha \mathcal{O}_1 + \sum_{i,k} \beta_i L_i^k \mathcal{O}_k + \sum_{i,j,k} \gamma_{ji} A_{ij}^k \mathcal{O}_k$$

$\gamma \geq 0$

$$\Longleftrightarrow \forall k \quad B_k = \alpha \delta_{1k} + \sum_i \beta_i L_i^k + \sum_{ij} \gamma_{ji} A_{ij}^k$$

variables

finite set of equalities.

To find best bound

linear

$$\max_{\alpha, \beta, \gamma} \alpha$$

linear equalities

undr

$\in \mathcal{L}_{2D}$

- $\forall k < k_{max}, \; B_k = \alpha \delta_{1k} + \dots$
- $\gamma \geq 0$

$\Rightarrow$ Semi definite program

PSD constraint

Examples : $\mathcal{O} = a^\dagger a$

- $\mathcal{L}(\rho) = -i\omega[a^\dagger a, \rho] + \kappa_1 \mathcal{D}[a]\rho + \kappa_2 \mathcal{D}[a^2-\alpha^2](\rho)$
- $\mathcal{L}(\rho) = \kappa_2 \mathcal{D}[a^2-\alpha^2](\rho)$

Open problem :

- no proof of convergence?
- $\mathrm{tr}\left[\rho_t \mathcal{O}\right]$ finite time?
- gap of $\mathcal{L}$???