Solving the Lindblad equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -i[H,\rho_t] + \sum_k L_k\rho_t L_k^\dagger - \frac{1}{2}\left\{L_k^\dagger L_k, \rho_t\right\}$$

Strategies and open problems

Antoine Tilloy

May 23, 2024

Quantic days, Annecy

What?

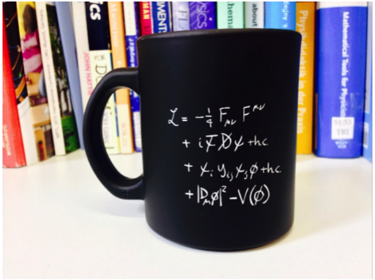
The most general **continuous Markovian** (time-local) equation for density matrices ρ:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho_t &= \mathcal{L}(\rho_t) \\ &= -i[H, \rho_t] + \sum_k L_k \rho_t L_k^\dagger - \frac{1}{2}\left\{L_k^\dagger L_k, \rho_t\right\}\end{aligned}$$

- $H = H^\dagger$
- $\{L_k\}$ are arbitrary

Lindblad equation as fundamental?

(So far) not a *fundamental* equation of Nature



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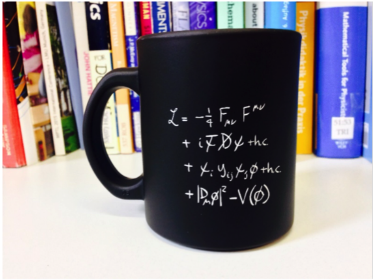
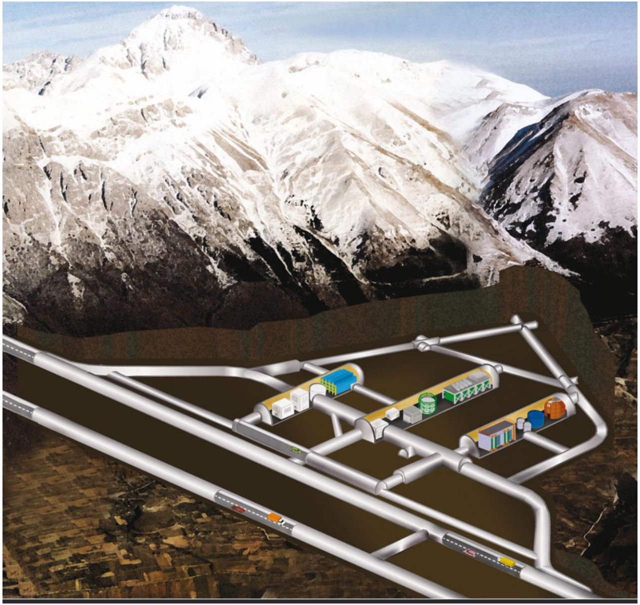


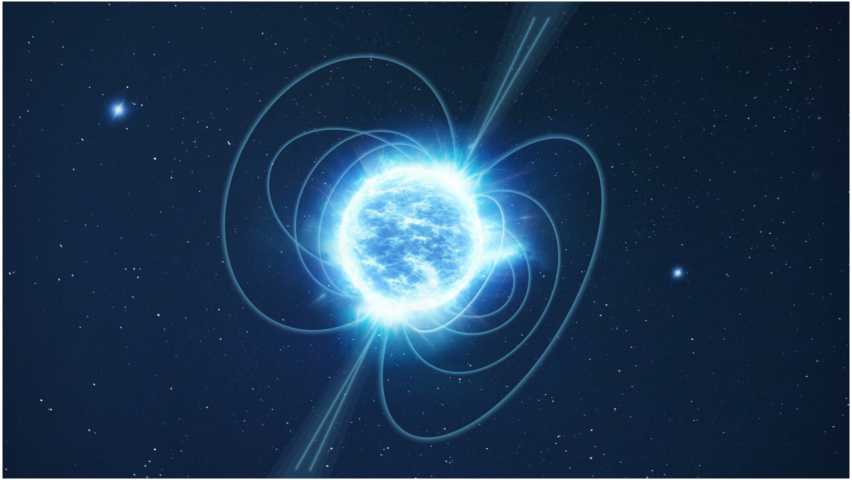
Locality + $L_k \neq 0 \quad \Longrightarrow \quad \mathrm{tr}[H\rho_t]$ increasing

Germanium does not emit X-rays spontaneously



credit Catalina Curceanu

Neutron stars do not stay hot

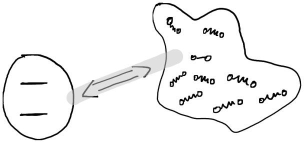


credit ESA

Derivations of the Lindblad equation

2 options

1. Coupling with a bosonic bath in some **Markov limit**



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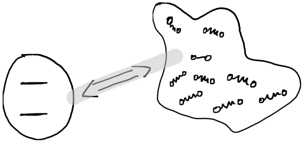
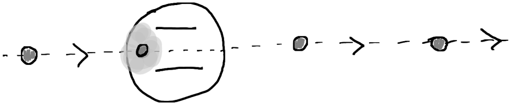


2. Repeated interactions in the **continuum limit** ← `easier`



Why?

- Simulate chips (or part of chips)

Why?

- Simulate chips (or part of chips)
- Solving field theories with continuous matrix product states

3 main questions

1. Real-time evolution ρ_t

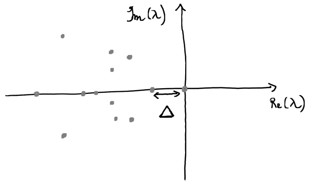
$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = \mathcal{L}(\rho_t)$$

2. Stationary state(s) ρ_0

$$\mathcal{L}(\rho_0) = 0$$

3. Spectral gap $\Delta := -\mathrm{Re}(\lambda_1)$

$$\mathcal{L}(\rho_1) = \lambda_1 \rho_1$$



Problem for Quantic

Large local Hilbert space

$$|\mathscr{H}| = N_c^n \text{ instead of } 2^n$$

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Large local Hilbert space

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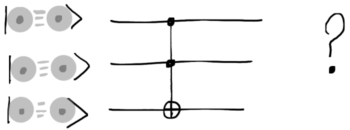
Simulating the Toffoli gate is already a hard problem



Goal

In the “good qubit” regime:

$$N_c^n \longrightarrow C \times (2 + \varepsilon)^n$$

with arbitrarily small error

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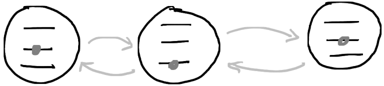
In the "good qubit" regime:

$$N_c^n \longrightarrow C \times (2+\varepsilon)^n$$

with arbitrarily small error

- Reasonable from complexity theory $\mathsf{BQP} \neq \mathsf{BPP}$
- With enough noise $N_c^n \longrightarrow \mathrm{poly}(n)$ `[for supremacy]`

The basics: “small” Hilbert space case



The one and only trick: $\mathcal{L}$ sparsity

Typical Lindbladian:

$$\mathcal{L}(\rho_t) := -i[H, \rho_t] + \sum_{k=1}^{r} L_k \rho_t L_k^\dagger - \frac{1}{2}\left\{L_k^\dagger L_k, \rho_t\right\}$$

2 sources of sparsity:

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$\implies$ See $\mathcal{L}$ as a *function* not a matrix

Real-time evolution – small $\mathscr{H}$

Find $t \mapsto \rho_t$ for

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Solutions

1. Exponentiate $\rho_t = \mathrm{e}^{t\mathcal{L}} \cdot \rho_{t=0}$ (with Krylov)
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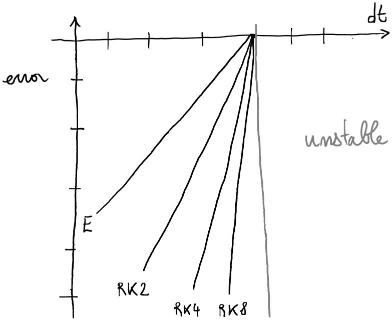
Exponentiation better for few points, ODE for full trajectory

Solving a (linear) ordinary differential equation

From Rémi's talk:

- Higher-order Runge-Kutta: numerically exact or blows up
- Low-order Rouchon: doesn't blow up



Stationary state

Finding ρ_0 such that $\mathcal{L}(\rho_0) = \rho_0$

Strategies:

1. Do real time evolution for a long time
2. Solve the linear problem $\mathcal{L}(\rho_0) = 0$
3. Find the lowest eigenvector of $\mathcal{L}$

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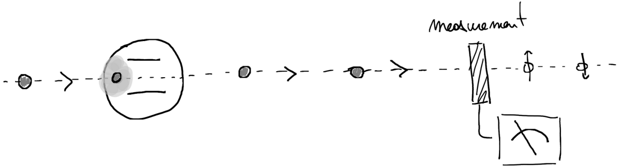
Cost: always rN^3 for iterative methods (N^6 for strictly exact)

Remark: Stochastic trajectory method

Quantum trajectory method

For ρ_t solution of Lindblad, $\exists$ $|\psi_t\rangle$ obeying **stochastic non-linear** equation such that:

$$\rho_t = \mathbb{E}\Big[|\psi_t\rangle\langle\psi_t|\Big]$$



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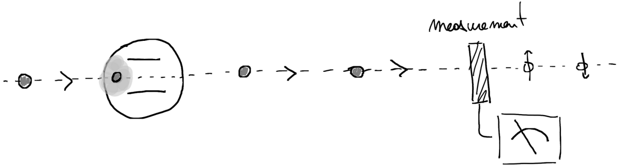


Example: photo-detection / homodyne-detection with $\eta = 1$

- Requires less memory
- Obliterates precision

Main interest: leveraging approximate algorithms made for pure states

With bosons

Bosonic Hilbert space:

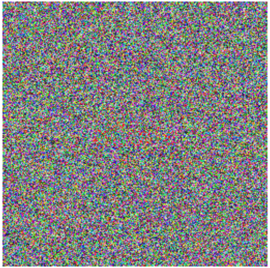
$$\mathscr{H} = L^2(\mathbb{R}) = \mathrm{span}\{a^{\dagger k}|0\rangle\}$$

Need some truncation / compression

Compression

No free lunch

There will always be situations in which $L^2(\mathbb{R})$ cannot efficiently be compressed



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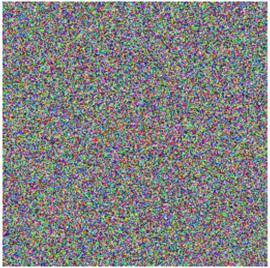

A compression method that is not bad at something, cannot be good at anything

Fock space truncation

Choose $a^{\dagger k}|0\rangle$ for $0 \leqslant k \leqslant N_c$
In real-space: Hermite functions

$$\psi_n(x) = H_n(x) \exp(-x^2/2)$$

Fock space truncation

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In real-space: Hermite functions

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Advantages:

- Error $\propto \mathrm{e}^{-CN_c}$, $\rightarrow$ **numerically exact**
- L_k often sparse in this basis

Fock space truncation

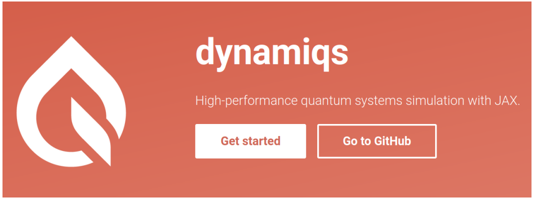
No free lunch:

$$N_c \propto |\alpha|^2$$

Good cat qubits require **larger** Hilbert space

State of the art: Dynamiqs

The previous tricks $+$ smart use of hardware $+$ autodiff $+$ batching $\rightarrow$ `dynamiqs`



Close to the numerical optimum

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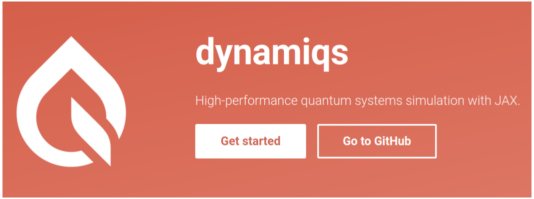

Close to the numerical optimum

`QuantumOptics.jl` also good (without seemless autodiff, batching, GPU)

Technical open problems

Exact stationary state

Find ρ_0 *exactly* at cost rN^3.
Currently:

- rN^3 approximate fixed-point algorithms (e.g. Krylov)
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Precise time-dependent real-time evolution

Find a method more precise than Runge-Kutta for time-dependent real-time dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = \mathcal{L}_t(\rho_t)$$

using the fact that $\mathcal{L}_t$ is *linear*

Summary for small Hilbert space

Lindblad for bosons = partial differential equations in $2n_{\text{bosons}} + 1$ dimensions

Good

- Solved **numerically exactly** at cost $rN^2 \propto N_c^{2n_{\text{bosons}}}$!
- It is implemented in user friendly libraries
- It is easy to implement manually to test

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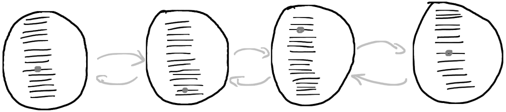
Bad

- For cats $N \sim (4|\alpha|^2)^{n_{\text{cats}}}$
- Compute and memory cost $\propto |2\alpha|^{4n_{\text{cats}}}$

Advanced: Large Hilbert space



Analytic solutions: lucky cases

Gaussian states

Equivalently:

- ρ is a Gaussian in configuration space
- ρ has a Gaussian Wigner function
- $\rho = \exp(\text{quadratic in } a, a^\dagger)$

$\implies$ ρ characterized by $\mathrm{tr}(a\rho)$, $\mathrm{tr}(aa\rho)$ and $\mathrm{tr}(a^\dagger a\rho)$

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Quadratic Lindbladian

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2. A_k at most linear in $a, a^\dagger$

$\mathcal{L}$ quadratic $\implies$ exactly solvable with Gaussian states

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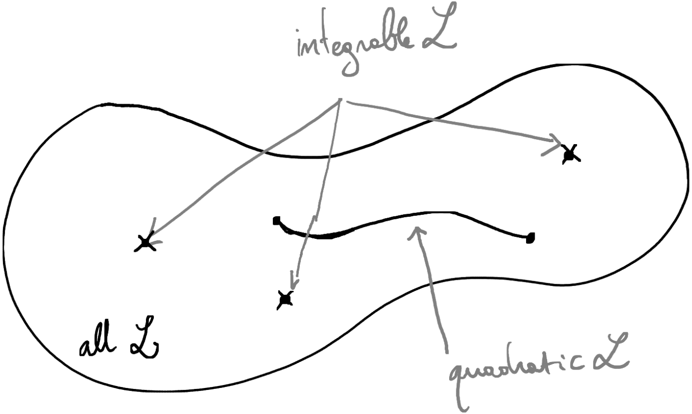
Small number of other exactly solvable (integrable) cases found by Tomaz Prosen

Exact solutions are too rare to be really useful



An idea?

Can't we find a better basis??

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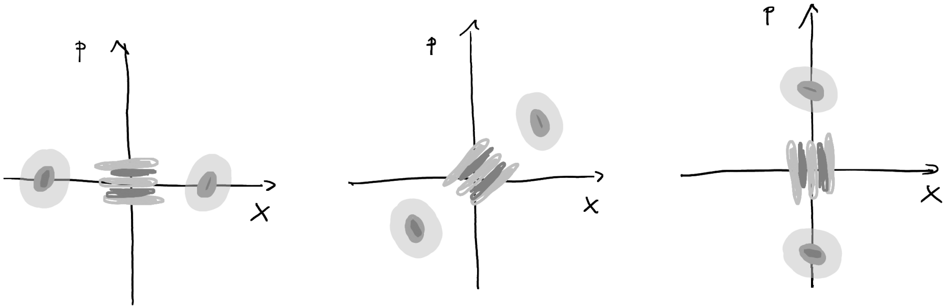
NO!

Linear stability should be abandonned

No "best" plane to approximate a sphere



Submanifold for local qubits

Many interesting subspaces $\mathcal{M} \subset \mathscr{H}$ are not vector spaces:

- coherent states
- Gaussian states
- ρ with fixed rank
- "Dynamical shifted Fock"

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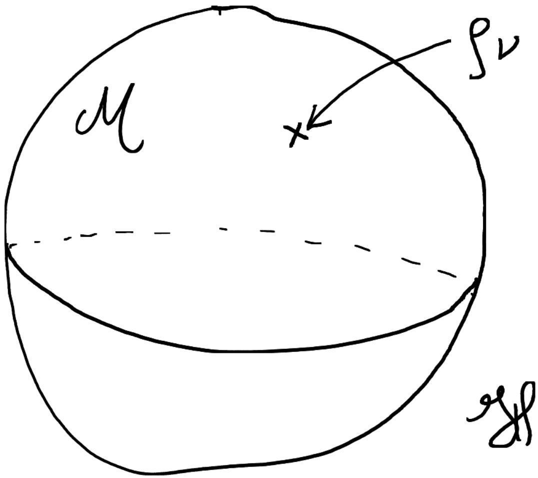
Example

$$\rho_\nu = |\psi_\nu\rangle\langle\psi_\nu| \quad \text{with} \quad \psi_\nu(x) = \exp(-\nu_1 x^2 - \nu_2 x^4)$$

Time-dependent variational principle (TDVP)



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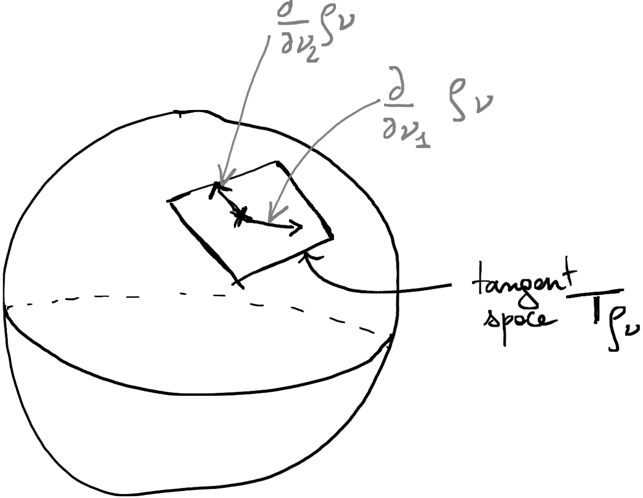

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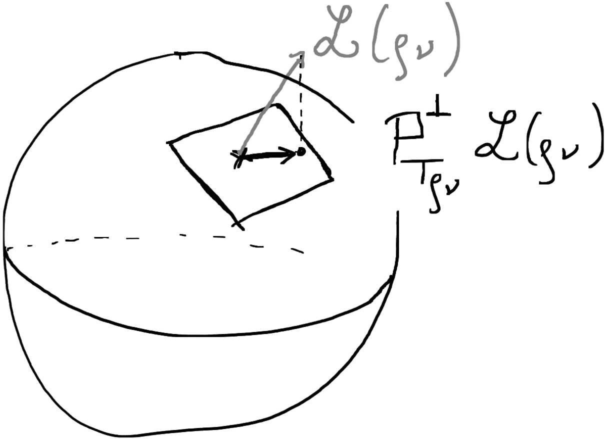

Time-dependent variational principle (TDVP)

McLachlan TDVP = minimize the error

$$\text{error} = \left\| \frac{\mathrm{d}}{\mathrm{d}t} \rho(\nu_t) - \mathcal{L}[\rho(\nu_t)] \right\|^2 \tag{1}$$

for convenience, take the Frobenius norm

$$\|A\| = \sqrt{\mathrm{tr}(A^\dagger A)} \tag{2}$$

TDVP

$$\frac{\partial}{\partial \frac{\mathrm{d}}{\mathrm{d}t}\nu_j} \left\| \frac{\mathrm{d}}{\mathrm{d}t}\rho(\nu_t) - \mathcal{L}[\rho(\nu_t)] \right\|^2 = 0$$

TDVP

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The time derivative of ρ is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(\nu_t) = \frac{\partial \rho}{\partial \nu_j}\frac{\mathrm{d}\nu_j}{\mathrm{d}t} = \frac{\partial \rho}{\partial \nu_j}\frac{\mathrm{d}}{\mathrm{d}t}\nu_j$$

thus

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$$\frac{d}{dt}\nu_j \operatorname{tr}\left[\left(\frac{\partial \rho}{\partial \nu_j}\right)^\dagger \frac{\partial \rho}{\partial \nu_k} + \text{h.c.}\right] - \operatorname{tr}\left[\left(\frac{\partial \rho}{\partial \nu_j}\right)^\dagger \mathcal{L}(\rho) + \text{h.c.}\right] = 0$$

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Hence the dynamics is $\frac{d}{dt}\nu_j = [g]^{-1}_{jk}\, u_k$ with

$$g_{jk} = \Re\mathrm{e}\left\{\operatorname{tr}\left[\left(\frac{\partial \rho}{\partial \nu_j}\right)^\dagger \frac{\partial \rho}{\partial \nu_k}\right]\right\}$$

$$u_k = \Re\mathrm{e}\left\{\operatorname{tr}\left[\left(\frac{\partial \rho}{\partial \nu_k}\right)^\dagger \mathcal{L}(\rho)\right]\right\}$$

Requirements

Solve the non-linear ODE

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with Runger-Kutta (or other)

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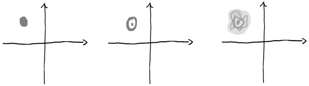
4 levels

- Exactly computable (Shifted Fock / MPS / Quantics) → `best`
- Well approximable (2d tensor networks)
- Can be directly sampled with Monte Carlo (some recurrent neural nets)
- Requires Markov Chain Monte Carlo for approximate sampling (generic neural nets)

Local strategy I: dynamical shifted Fock

The shifted Fock ladder (vector space):

$$\mathcal{M}_\alpha^{N_c} = \text{span}\{a^{\dagger k}|\alpha\rangle,\ \ k \leqslant N_c\} = \text{span}\{D(\alpha)|k\rangle,\ \ k \leqslant N_c\}$$

$=$ 

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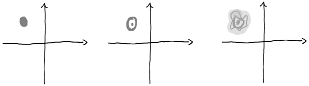
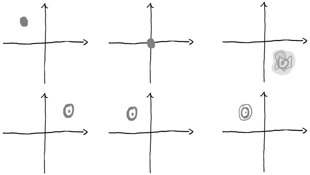
$$=$$ 

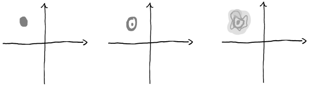
The "dynamical" shifted Fock ladder (manifold):

$$\mathcal{M}^{N_c} = \{|\psi\rangle \in \mathcal{M}_\alpha,\ \alpha \in \mathbb{C}\} =$$ 

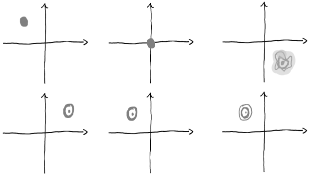
Local strategy I: dynamical shifted Fock

The shifted Fock ladder (vector space):

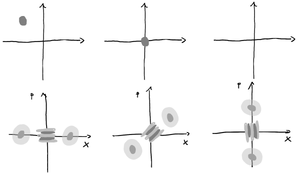
$$\mathcal{M}_{\alpha}^{N_c} = \mathrm{span}\{a^{\dagger k}|\alpha\rangle, \quad k \leqslant N_c\} = \mathrm{span}\{D(\alpha)|k\rangle, \quad k \leqslant N_c\}$$

$$=$$ 

The "dynamical" shifted Fock ladder (manifold):

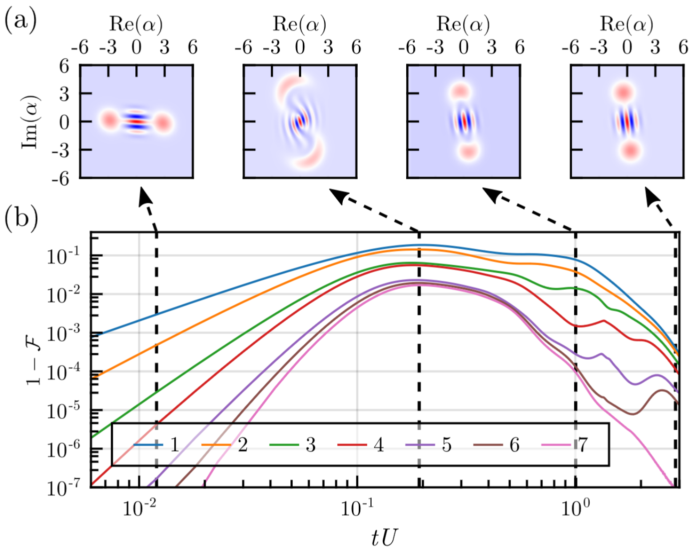
$$\mathcal{M}^{N_c} = \{|\psi\rangle \in \mathcal{M}_{\alpha}, \ \alpha \in \mathbb{C}\} =$$ 

The multi "dynamical" shifted Fock ladder (manifold):

$$\mathcal{M}_{(n)}^{N_c} = \left\{ |\psi\rangle = \sum_{i=1}^{n} |\psi_i\rangle, \ |\psi_i\rangle \in \mathcal{M} \right\} =$$ 

Dynamical shifted Fock

Proposed by Schlegel, Minganti, Savona in 2306.13708



Dynamical shifted Fock: pros and cons

The Good

- Good intuition for compression
- Works as intended $N_c \to 2 + \varepsilon$

The Bad

- Sparsity lost
- Potential ill-conditioning (fixable)
- Cannot destroy the blobs too much

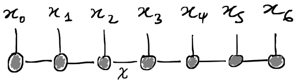
Likely not faster for 1 mode

Quantics

Idea of Adrien Moulinas and Xavier Waintal

Write ψ as an MPS in binary:

$$\psi(x_0.x_1x_2x_3x_4x_5x_6) = \langle A(0)_{x_0}|A(1)_{x_1}A(2)_{x_2}A(3)_{x_3}A(4)_{x_4}A(5)_{x_5}|A(6)_{x_6}\rangle$$

$=$ 

with

- $A(0)_0, A(0)_1$ and $A(6)_0, A(6)_1$ vectors $\in \mathbb{C}^\chi$
- $A(1)_0, A(1)_1, \cdots A(5)_0, A(5)_1$ matrices $\in \mathbb{C}^\chi \otimes \mathbb{C}^\chi$

Quantics: pros and cons

Still to be tested, but:

The Good (promising):

- Leverages powerful tensor network routines (could be fast!)
- Can resolve arbitrarily small scales

The Bad (worrying):

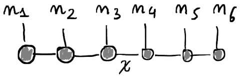
- Not super clear when it should work intuitively
- Entanglement in scale??

Matrix product state like approaches

Mainly 3 options (tried at Eviden on slightly different problems)

1. Matrix Product States for stochastic ψ

$$\psi(n_1, \cdots, n_6) =$$



Matrix product state like approaches

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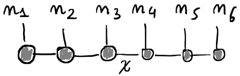
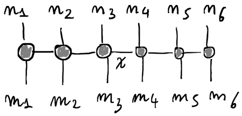


2. Matrix Product Operators for ρ

$$\rho(n_1, \cdots, n_6, m_1, \cdots, m_6) =$$

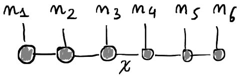


Matrix product state like approaches

Mainly 3 options (tried at Eviden on slightly different problems)

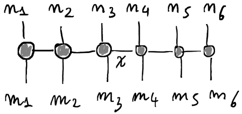
1. **1.** Matrix Product States for stochastic ψ

$$\psi(n_1, \cdots, n_6) =$$



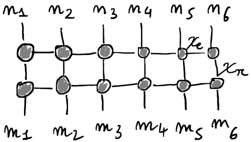
2. **2.** Matrix Product Operators for ρ

$$\rho(n_1, \cdots, n_6, m_1, \cdots, m_6) =$$



3. **3.** Matrix Product Density Operator for ρ (aka *purification*)

$$\rho(n_1, \cdots, n_6, m_1, \cdots, m_6) =$$



MPS approaches: intuition

Hope: $\chi \propto (2+\varepsilon)^n$ (not much inter-cat entanglement)

- No local physical space compression
- Asymptotic cost $N_c^4 * (2+\varepsilon)^n$

No gain on $\leqslant 2$ modes

Strategy

Small exact simulations: memory cost N_c^{2n}

- Mostly Solved
- Few open problems (very high order Rouchon, cheaper exact stationary states)
- No huge gain expected unless ultra-stiff
- Well implemented

Large simulations: target is memory cost $C \times (2 + \varepsilon)^{2n}$

- Local compression way (Dynamically shifted Fock, Quantics)
- Global compression way (MPS,MPO,MPDO)
- Nothing well implemented yet

Do not ask for performance increase at the beginning for small systems!