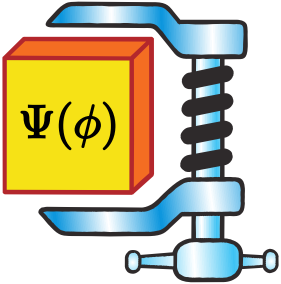
The Sinh-Gordon model with relativistic continuous matrix product states



Antoine Tilloy

April 26th, 2023

IFT, Madrid

MINES PARIS | PSL

ENS | PSL

Inria

erc

Objectives of the day

Baby steps of a program to solve relativistic QFT without discretizing:

1. A variational method: relativistic continuous matrix product states
   `arXiv:2102.07733` and `arXiv:2102.07741`

$$|Q,R\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q\otimes \mathbb{1} + R\otimes a^\dagger(x)\right]\right\}|0\rangle_a$$

2. A subtle $1+1d$ field theory: the Sinh-Gordon model – near "self-duality"
   `arXiv:2209.05341`

$$H_{\mathrm{ShG}}(\beta) = \int \mathrm{d}x\, \frac{:\pi^2:}{2} + \frac{:(\nabla\phi)^2:}{2} + \frac{m^2}{\beta^2} :\cosh(\beta\phi):$$

Quantum field theory: general objective

Long term goal

Find methods to solve "real world" quantum field theories (even without structure) to good (machine?) precision

Go beyond the currently leading approaches

1. Perturbation theory $\leftarrow$ need resummation / expensive large orders
2. Lattice Monte Carlo $\leftarrow$ need discretization / slow convergence of error / sign

Quantum field theory: general objective

Long term goal

Find methods to solve "real world" quantum field theories (even without structure) to good (machine?) precision

Go beyond the currently leading approaches

1. Perturbation theory ← need resummation / expensive large orders
2. Lattice Monte Carlo ← need discretization / slow convergence of error / sign

3 **promising alternatives**

1. Bootstrap / SDP relaxations / Sum of Squares
2. Renormalization group ← functional or tensor network RG
3. **Variational method** ← Hamiltonian truncation or tensor network states

Similarity between relativistic and critical models

- A critical model is scale invariant in the IR
- A relativistic QFT is scale invariant in the UV

Similarity between relativistic and critical models

- A critical model is scale invariant in the IR
- A relativistic QFT is scale invariant in the UV

This implies divergence of the entanglement entropy in $1+1d$

$$S \propto c \log(\Lambda)$$

$\Longrightarrow$ Continuum limit requires finite entanglement scaling

double scaling (UV and IR) by Vanhecke, Verstraete, Van Acoleyen `2104.10564`

The variational method

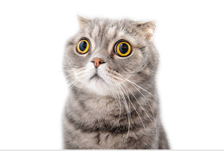
directly in the continuum

Idea of the variational method

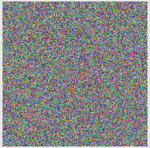
Variational method for ground state search

1. Guess a manifold $\mathcal{M} \subset \mathscr{H}$ with few parameters ν *i.e.* $\dim\mathcal{M} \ll \dim\mathscr{H}$
2. Tune ν to minimize energy $\nu = \text{argmin}_{\nu\in\mathcal{M}} \frac{\langle\nu|H|\nu\rangle}{\langle\nu|\nu\rangle}$ and get $|\text{ground state}\rangle \simeq |\nu\rangle$

Reason for compression (classical)



cat image



“typical” image

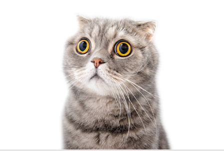
atypical $\Longrightarrow$ compressible

Idea of the variational method

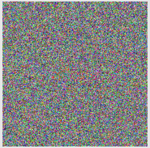
Variational method for ground state search

1. Guess a manifold $\mathcal{M} \subset \mathscr{H}$ with few parameters ν *i.e.* $\dim \mathcal{M} \ll \dim \mathscr{H}$
2. Tune ν to minimize energy $\nu = \operatorname{argmin}_{\nu \in \mathcal{M}} \frac{\langle \nu | H | \nu \rangle}{\langle \nu | \nu \rangle}$ and get $|\text{ground state}\rangle \simeq |\nu\rangle$

Reason for compression (classical)



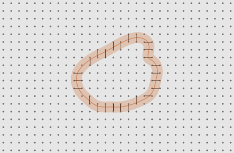
cat image



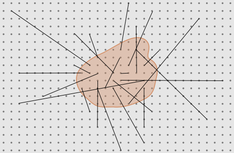
"typical" image

atypical $\implies$ compressible

Reason for compression (quantum)



low energy state



random state

area law = atypical $\implies$ compressible

Feynman's criticism

Difficulties in Applying the Variational Principle to Quantum Field Theories[1]

so I tried to do something along these lines with quantum chromodynamics. So I'm talking on the subject of the application of the variational principle to field theoretic problems, but in particular to quantum chromodynamics.

I'm going to give away what I want to say, which is that I didn't get anywhere! I got very discouraged and I think I can see why the variational principle is not very useful. So I want to take, for the sake of argument, a very strong view – which is stronger than I really believe – and argue that it is no damn good at all!

Feynman's requirement in a nutshell

1. **Extensive parameterization**
   Number of parameters $\propto L^{\alpha}$ at most for system size L (not $\propto \mathrm{e}^{L}$)
2. **Computable expectation values**
   ψ known $\implies \langle \mathcal{O}(x)\mathcal{O}(y)\rangle_{\psi}$ computable
3. **Not oversensitive to the UV**
   no runaway minimization where higher and higher momenta get fitted

Elegantly swallowing the bullet

Example: naive Hamiltonian truncation

With an IR cutoff L, momenta are discrete. Take as submanifold $\mathscr{M}$ the **vector space** spanned by:

$$|k_1, k_2, \cdots, k_r\rangle = a^\dagger_{k_1} a^\dagger_{k_2} \cdots a^\dagger_{k_r} |0\rangle_a$$

such that $\langle k_1 k_2 \cdots k_r | H | k_1 k_2 \cdots k_r \rangle \leqslant E_{\text{trunc}} \;\rightarrow\;$ finite dimensional

Breaks **extensiveness**

- number of parameters $\propto \mathrm{e}^{L \times E_{\text{trunc}}}$
- error $\propto E_{\text{trunc}}^{-3}$ (with renormalization refinements)

still good results, see *e.g.* Rychkov & Vitale for ϕ^4_2 `arXiv:1412.3460`

Intuition: put 2 ingredients together

1- Extensive parameterization and 2- Computable expectation values

Realized by **tensor network states** on the lattice
e.g. in $1+1$ dimensions: Matrix Product states (MPS)

$$|\psi(A)\rangle := \sum_{i_1, i_2, \ldots, i_N} \operatorname{tr}\left[A_{i_1} A_{i_2} \cdots A_{i_N}\right] |i_1, i_2, \ldots, i_N\rangle$$

where A_i are matrices $\in \mathcal{M}_D(\mathbb{C})$

3- Not oversensitive to the UV

Realized by **Hamiltonian truncation**, *i.e.* working in the Fock basis

$$|k_1, k_2, \cdots, k_r\rangle = a^\dagger_{k_1} a^\dagger_{k_2} \cdots a^\dagger_{k_r} |0\rangle_a$$

Intuition: put 2 ingredients together

1- Extensive parameterization and 2- Computable expectation values

Realized by **tensor network states** on the lattice
e.g. in $1+1$ dimensions: Matrix Product states (MPS)

$$|\psi(A)\rangle := \sum_{i_1, i_2, \dots, i_N} \text{tr}\left[A_{i_1} A_{i_2} \cdots A_{i_N}\right] |i_1, i_2, \dots, i_N\rangle$$

where A_i are matrices $\in \mathcal{M}_D(\mathbb{C})$

3- Not oversensitive to the UV

Realized by **Hamiltonian truncation**, *i.e.* working in the Fock basis

$$|k_1, k_2, \cdots, k_r\rangle = a^\dagger_{k_1} a^\dagger_{k_2} \cdots a^\dagger_{k_r} |0\rangle_a$$

Strategy: MPS $\xrightarrow[\text{continuum limit}]{}$ CMPS (2010) $\xrightarrow[\text{change of basis}]{}$ RCMPS (2021)

Relativistic continuous matrix product states

Definition

RCMPSs are a manifold of states parameterized by 2 $(D \times D)$ matrices Q, R

$$|Q, R\rangle = \mathrm{tr}\left\{\mathcal{P} \exp\left[\int \mathrm{d}x \, Q \otimes \mathbb{1} + R \otimes a^\dagger(x)\right]\right\} |0\rangle_a$$

with

- $a(x) = \frac{1}{2\pi} \int \mathrm{d}k \, \mathrm{e}^{ikx} a_k$ where $a_k = \frac{1}{\sqrt{2}}\left(\sqrt{\omega_k}\,\hat{\phi}(k) + i\frac{\hat{\pi}(k)}{\sqrt{\omega_k}}\right)$ free modes
- trace taken over $\mathbb{C}^D$
- $\mathcal{P}$ path-ordering exponential

Relativistic continuous matrix product states

Definition

RCMPSs are a manifold of states parameterized by 2 $(D \times D)$ matrices Q, R

$$|Q, R\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q \otimes \mathbb{1} + R \otimes a^\dagger(x)\right]\right\}|0\rangle_a$$

with

- $a(x) = \frac{1}{2\pi}\int \mathrm{d}k\, \mathrm{e}^{ikx} a_k$ where $a_k = \frac{1}{\sqrt{2}}\left(\sqrt{\omega_k}\,\hat{\phi}(k) + i\frac{\hat{\pi}(k)}{\sqrt{\omega_k}}\right)$ free modes
- trace taken over $\mathbb{C}^D$
- $\mathcal{P}$ path-ordering exponential

1. Continuum MPS (CMPS) $+$ Gaussian disentangling (cMERA) $|0\rangle_\psi \to |0\rangle_a$ (noted by Niloofar Vardian in 2208.14827)
2. "Non-commutative" free particle coherent state

Basic properties of RCMPS

$$|Q, R\rangle = \mathrm{tr}\left\{\mathcal{P} \exp\left[\int \mathrm{d}x\, Q \otimes \mathbb{1} + R \otimes a^{\dagger}(x)\right]\right\} |0\rangle_a$$

Basic properties of RCMPS

$$|Q, R\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q \otimes \mathbb{1} + R \otimes a^{\dagger}(x)\right]\right\}|0\rangle_a$$

Feynman's checklist:

1. **Extensive** because of $\mathcal{P}\exp\int$

Basic properties of RCMPS

$$|Q,R\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q\otimes \mathbb{1} + R\otimes a^\dagger(x)\right]\right\}|0\rangle_a$$

Feynman's checklist:

1. **Extensive** because of $\mathcal{P}\exp\int$
2. Obervables **computable** at cost D^3 (non trivial!) requires $[a(x), a^\dagger(y)] = \delta(x-y)$

Basic properties of RCMPS

$$|Q, R\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\left[\int \mathrm{d}x\, Q \otimes \mathbb{1} + R \otimes a^\dagger(x)\right]\right\}|0\rangle_a$$

Feynman's checklist:

1. **Extensive** because of $\mathcal{P}\exp\int$
2. Obervables **computable** at cost D^3 (non trivial!)
   requires $[a(x), a^\dagger(y)] = \delta(x - y)$
3. **No UV problems**
   $|0, 0\rangle = |0\rangle_a$ is the ground state of H_0 hence exact CFT UV fixed point
   $\langle Q, R| : P(\phi) : |Q, R\rangle$ is finite for all Q, R (not trivial!)
   Entanglement Entropy in $a(x)$ basis is **finite**

The variational algorithm

Optimization

Compute $e_0 = \langle Q, R|h|Q, R\rangle$ and $\nabla_{Q,R} e_0$
Minimize e_0 with (geometric improvements of) gradient descent

The variational algorithm

Optimization

Compute $e_0 = \langle Q, R|h|Q, R\rangle$ and $\nabla_{Q,R} e_0$
Minimize e_0 with (geometric improvements of) gradient descent

Computations of e_0 and ∇e_0 in a nutshell:

1. $V_b = \langle :e^{b\phi(x)}:\rangle_{QR}$ computable by solving an ODE with cost $\propto D^3$
2. $\langle :\phi^n:\rangle_{QR}$ computable doing $\partial_b^n V_b\Big|_{b=0} \rightarrow \propto D^3$
3. $e_0 = \langle h\rangle_{QR}$ computable by summing such terms at cost $D^3 \rightarrow \propto D^3$
4. ∇e_0 computable by solving the adjoint ODE (backpropagation) $\rightarrow \propto D^3$

The variational algorithm

Optimization

Compute $e_0 = \langle Q, R|h|Q, R\rangle$ and $\nabla_{Q,R} e_0$
Minimize e_0 with (geometric improvements of) gradient descent

Computations of e_0 and ∇e_0 in a nutshell:

1. $V_b = \langle :e^{b\phi(x)}: \rangle_{QR}$ computable by solving an ODE with cost $\propto D^3$
2. $\langle :\phi^n: \rangle_{QR}$ computable doing $\partial_b^n V_b \Big|_{b=0} \rightarrow \propto D^3$
3. $e_0 = \langle h \rangle_{QR}$ computable by summing such terms at cost $D^3 \rightarrow \propto D^3$
4. ∇e_0 computable by solving the adjoint ODE (backpropagation) $\rightarrow \propto D^3$

Functioning `Julia` implementation. `OptimKit.jl` to solve the Riemannian minimization, `KrylovKit.jl` to solve fixed point equations, `DifferentialEquations.jl` (`Vern7` solver) to solve ODE. Soon `Rcmps.jl`?

Using the optimized state

After optimization: $|Q, R\rangle \simeq |0\rangle_{\text{int.}}$ with $\langle Q, R| \, \hat{h} \, |Q, R\rangle = e_0 + \varepsilon$

Using the optimized state

After optimization: $|Q, R\rangle \simeq |0\rangle_{\text{int.}}$ with $\langle Q, R| \, \hat{h} \, |Q, R\rangle = e_0 + \varepsilon$

This gives:

- All equal-time N-point functions

$$\langle \phi(x_1)\, \phi(x_2) \cdots \phi(x_n) \rangle \simeq \langle Q, R| \, \hat{\phi}(x_1)\, \hat{\phi}(x_2) \cdots \hat{\phi}(x_n) \, |Q, R\rangle$$

  at cost D^3 by solving coupled linear ODEs

Using the optimized state

After optimization: $|Q, R\rangle \simeq |0\rangle_{\text{int.}}$ with $\langle Q, R| \, \hat{h} \, |Q, R\rangle = e_0 + \varepsilon$

This gives:

- All equal-time N-point functions

$$\langle \phi(x_1) \, \phi(x_2) \cdots \phi(x_n) \rangle \simeq \langle Q, R| \, \hat{\phi}(x_1) \, \hat{\phi}(x_2) \cdots \hat{\phi}(x_n) \, |Q, R\rangle$$

at cost D^3 by solving coupled linear ODEs

- In particular *all* Euclidean 2-point functions $\implies$ spectral function

$$\langle \phi(x)\phi(0)\rangle = \int_0^{+\infty} \mathrm{d}\mu \, \mu\rho(\mu) K_0(\mu x)$$

Using the optimized state

After optimization: $|Q,R\rangle \simeq |0\rangle_{\text{int.}}$ with $\langle Q,R|\,\hat{h}\,|Q,R\rangle = e_0 + \varepsilon$

This gives:

- All equal-time N-point functions

$$\langle \phi(x_1)\,\phi(x_2)\cdots\phi(x_n)\rangle \simeq \langle Q,R|\,\hat{\phi}(x_1)\,\hat{\phi}(x_2)\cdots\hat{\phi}(x_n)\,|Q,R\rangle$$

  at cost D^3 by solving coupled linear ODEs

- In particular *all* Euclidean 2-point functions $\implies$ spectral function

$$\langle \phi(x)\phi(0)\rangle = \int_0^{+\infty} d\mu\, \mu\rho(\mu) K_0(\mu x)$$

One could get more observable with real-time evolution and tangent space methods, but is it needed in principle?

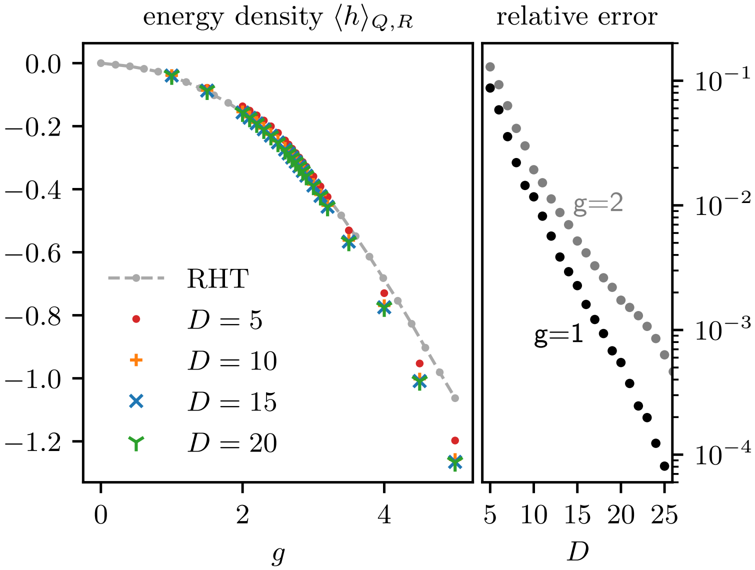
Hamiltonian definition of ϕ^4_2

Renormalized ϕ^4_2 theory

$$H = \int \mathrm{d}x \, \frac{:\pi^2:_m}{2} + \frac{:(\nabla\phi)^2:_m}{2} + \frac{m^2}{2} :\phi^2:_m + g :\phi^4:_m$$

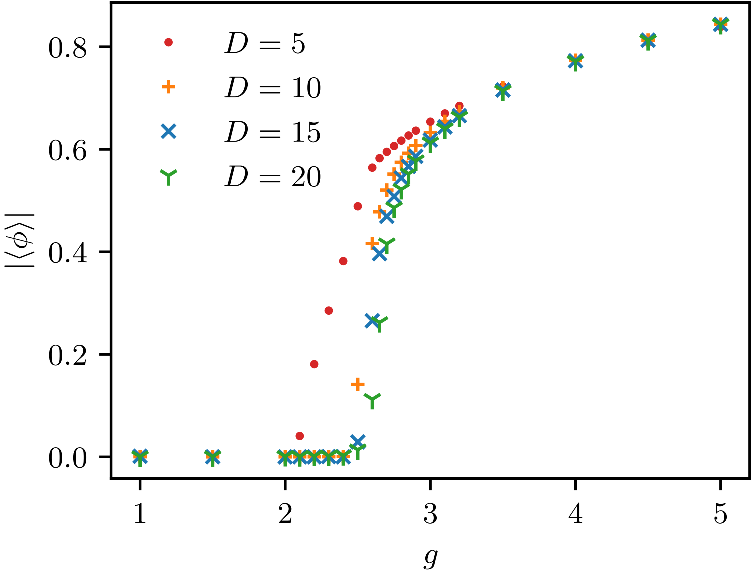
1. Rigorously defined relativistic QFT without cutoff (Wightman QFT)
2. Vacuum energy density ε_0 finite for all g
3. Difficult to solve unless $g \ll m^2$ (perturbation theory)
4. Phase transition around $f_c = \frac{g}{4m^2} = 11$ i.e. $g \simeq 2.7$ in mass units

Results: ϕ^4_2 energy density



New: D can now be pushed to 32 or even 64 with some effort

Results: ϕ_2^4 – field expectation value $\langle\phi\rangle$



New: the mass can be fitted from 2-point function and agrees with RHT to 10^{-3}

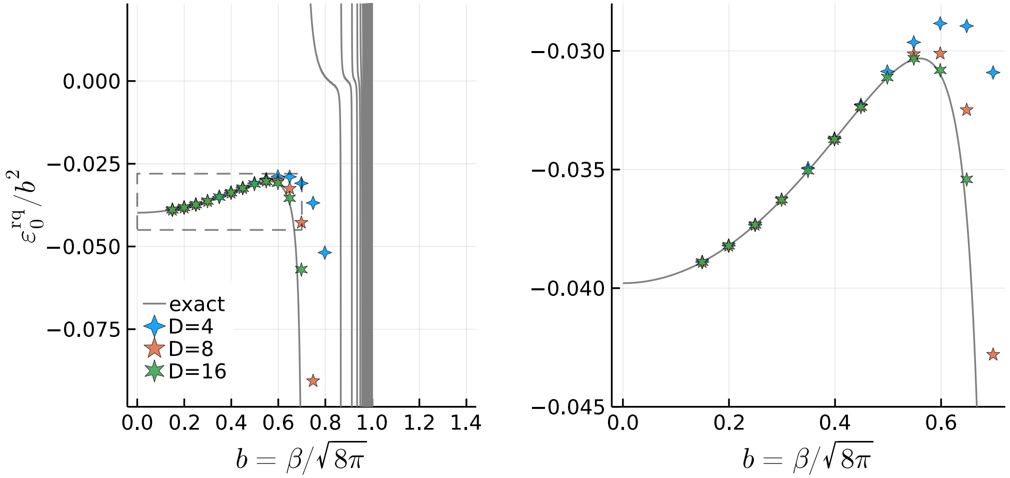
Hamiltonian definition of Sine-Gordon theory

Renormalized $\cos(\beta\phi)$ theory

$$H = \int dx \, \frac{:\pi^2:_m}{2} + \frac{:(\nabla\phi)^2:_m}{2} - \frac{m^2}{\beta^2} :\cos(\beta\phi):_m$$

1. Well defined for $b = \beta/\sqrt{8\pi} < 1/\sqrt{2}$
2. Ground energy density $\to -\infty$ for $b \to 1/\sqrt{2}$ but renormalizable until $b = 1$
3. Vertex operators, mass spectrum, and (renormalized) energy known exactly

Results: $\cos(\beta\phi)$ (rescaled) energy density



Fits arbitrarily well for $b \in [0, 1/\sqrt{2}[$, collapses to $-\infty$ for b larger

Numerically refines Coleman's argument from $b = 1$ to $b = 1/\sqrt{2} + \epsilon(D)$

The Sinh-Gordon model

the surprisingly subtle $\cosh(\beta\phi)$ potential

The Sinh-Gordon model

An exactly solvable model that is surprisingly subtle. Two recent studies

- Könik, Lájer, and Mussardo [KLM] arXiv:2007.00154
- Bernard and LeClair [BLC] arXiv:2112.05490

[Equal-time quantization] Hamiltonian definition

$$H_{\mathrm{ShG}}(\beta) = \int \mathrm{d}x \, \frac{:\pi^2:_m}{2} + \frac{:(\nabla\phi)^2:_m}{2} + \frac{m^2}{\beta^2} :\cosh(\beta\phi):_m$$

[Radial quantization] Dilation operator definition

$$D_{\mathrm{ShG}}(b) = D_0 + \mu \int_C \mathrm{d}z \, [\mathcal{V}_b(z, z^*) + \mathcal{V}_{-b}(z, z^*)]$$

Equivalent formulations with $b = \beta/\sqrt{8\pi}$ and $\mu = \frac{m^{2+2b^2}}{2^{4+2b^2}\pi b^2} e^{2b^2\gamma_E}$

The Sinh-Gordon model: puzzles

$$H_{\text{ShG}}(\beta) = \int \mathrm{d}x \, \frac{:\pi^2:_m}{2} + \frac{:(\nabla\phi)^2:_m}{2} + \frac{m^2}{\beta^2} :\cosh(\beta\phi):_m$$

Should be easy:

1. Intuitively should always make sense ($\cosh(\beta\phi)$ always relevant)
2. S-matrix, energy density, masses, vertex operators, “exactly” known
3. Apparent $b \to b^{-1}$ duality with normalized coupling $b = \beta/\sqrt{8\pi}$

The Sinh-Gordon model: puzzles

$$H_{\text{ShG}}(\beta) = \int \mathrm{d}x \, \frac{:\pi^2:_m}{2} + \frac{:(\nabla\phi)^2:_m}{2} + \frac{m^2}{\beta^2} :\cosh(\beta\phi):_m$$

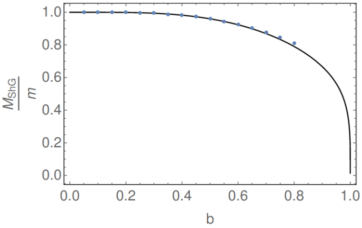
Should be easy:

1. Intuitively should always make sense ($\cosh(\beta\phi)$ always relevant)
2. S-matrix, energy density, masses, vertex operators, “exactly” known
3. Apparent $b \to b^{-1}$ duality with normalized coupling $b = \beta/\sqrt{8\pi}$

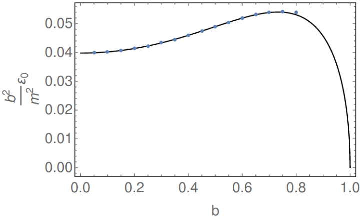
But unclear what the domain of validity of the formula is...

- Mass vanishes at $b = 1$ and likely stays at 0 [KLM and BLC]
- Likely no self-duality
- Could the exact formula break down before $b = 1$?
- Very hard to check numerically (despite thorough exploration of KLM)

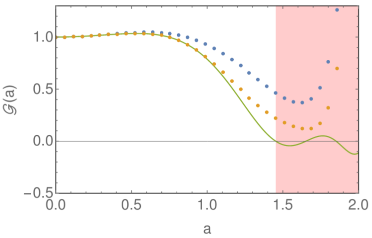
Some KLM results



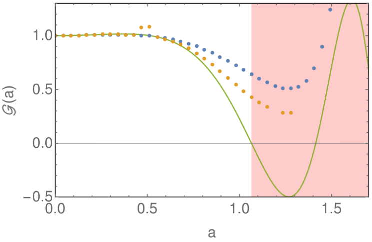
(a) Measured versus theoretical masses, normalized with respect to m.



(b) Measured versus theoretical vacuum energies, normalized with respect to $\frac{m^2}{b^2}$.

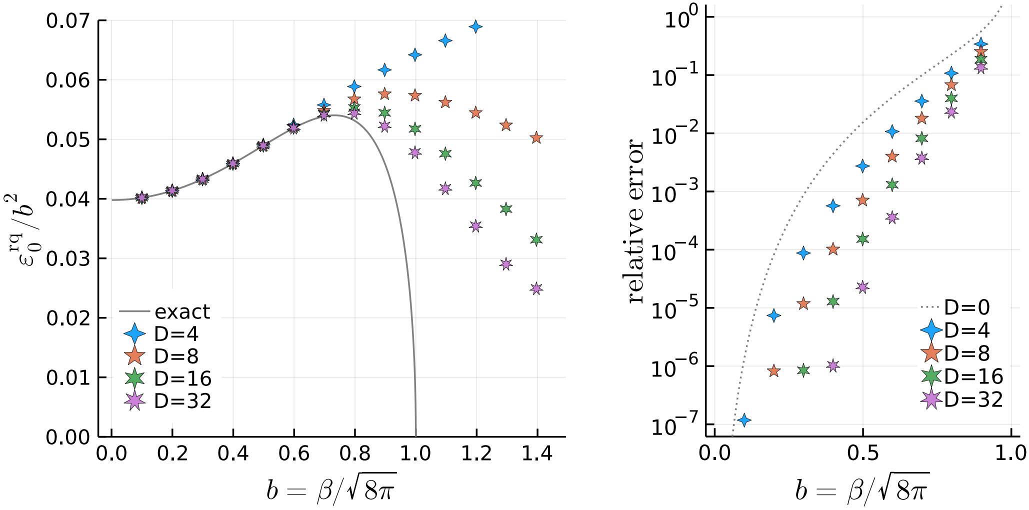


(a) Expectation values for $b = \frac{2}{\sqrt{8\pi}} \approx 0.4$.



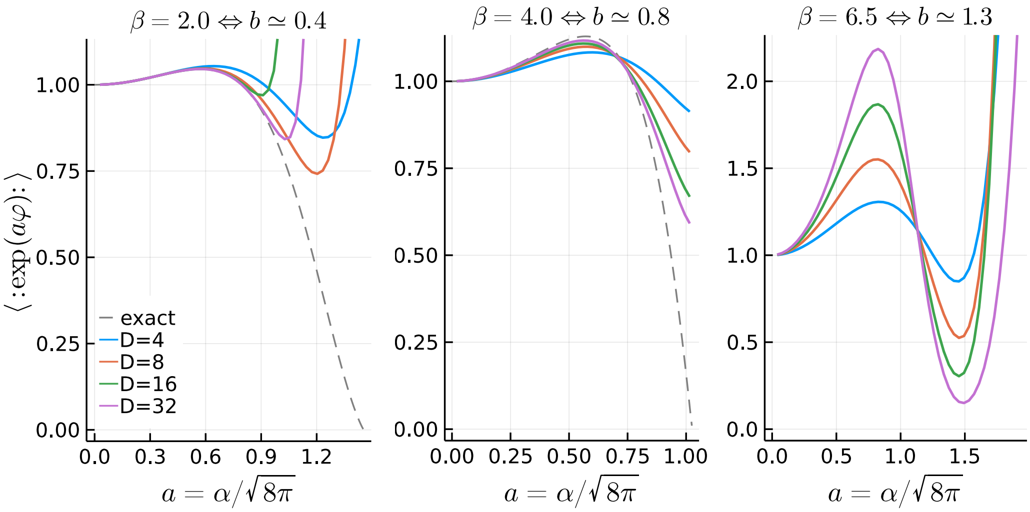
(b) Expectation values for $b = \frac{4}{\sqrt{8\pi}} \approx 0.8$.

Results: (rescaled) energy density

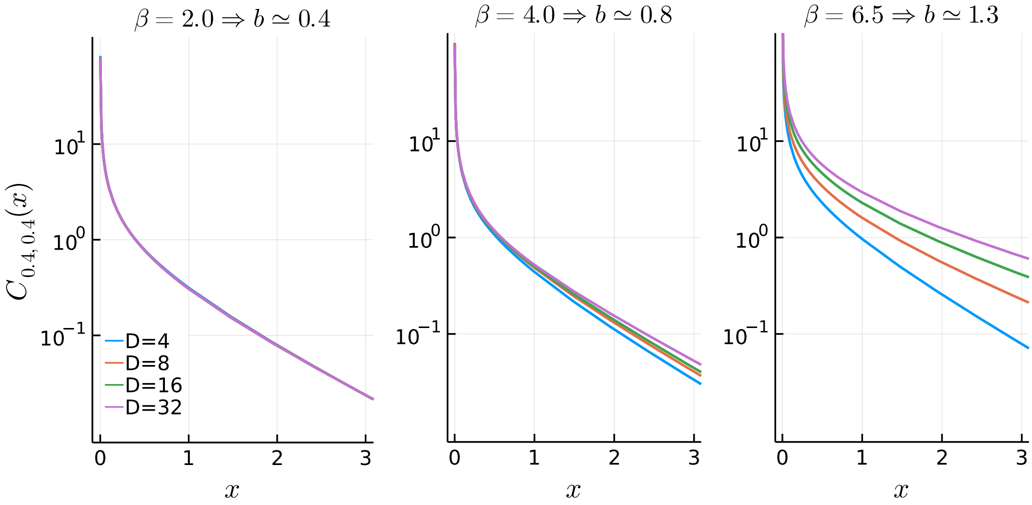


Results: vertex operators $\langle :e^{a\varphi}: \rangle$

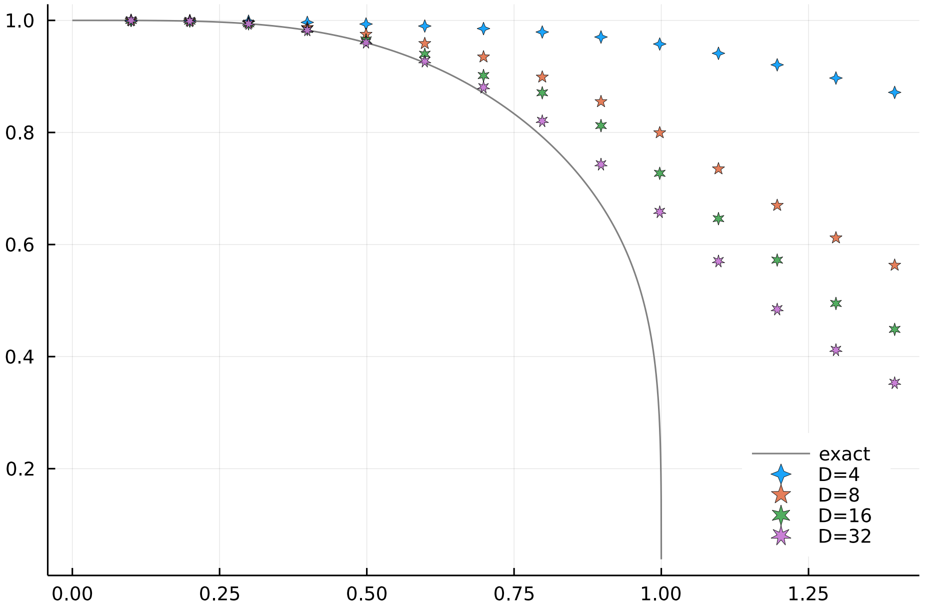
Known exactly from FLZZ formula up to $a = (b + b^{-1})/2$ (Seiberg bound)



Results: 2-point func $\langle :e^{a\varphi(x)}::e^{a\varphi(0)}:\rangle - \langle :e^{a\varphi(x)}:\rangle\langle :e^{a\varphi(0)}:\rangle$



New: mass gap from spectral function fit



Discussion and open problems

Understanding expressiveness of RCMPS

Standard Entanglement Entropy

Defined for "standard" locality

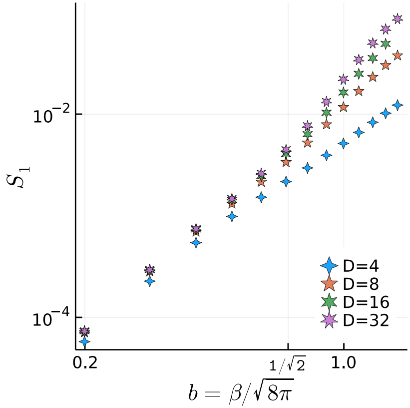
$$\rho_{\geqslant 0} = \int \prod_{x \leqslant 0} d\phi(x) \langle \phi | \Psi \rangle \langle \Psi | \phi \rangle$$

Gives $S_1 = -\mathrm{tr}\left(\rho_{\geqslant 0} \log \rho_{\geqslant 0}\right) \propto \log(\Lambda)$

Exotic Entanglement Entropy

Defined for RCMPS notion of locality
trace over $a^\dagger(x_1) \cdots a^\dagger(x_n)|0\rangle_m$ for $x_k \leqslant 0$

Gives $S_1 = O(1)$ (numerically)



EEE is finite at least for
$b \leqslant 1/\sqrt{2}$

Sinh-Gordon theory: what do we know?

Still uncertainty, following KLM, BLC, and the present study...
Would benefit from extrapolations in D!

My estimates:

1. 99% chance: Hamiltonian H has no self-duality $b \to b^{-1}$
2. 80% chance: Any reasonable definition of the model is massless for $b \geqslant 1$
3. 70% chance: Energy formula correct for $b \in [0, 1]$, and $e_0 = 0$ for $b \geqslant 1$.
4. 50% chance: FLZZ formula correct for *all* $a \geqslant (b + b^{-1})/2$
5. 50% chance: The model makes sense, without renormalization, for $b \geqslant 1$

Sinh-Gordon theory: what do we know?

Still uncertainty, following KLM, BLC, and the present study...
Would benefit from extrapolations in D!

My estimates:

1. 99% chance: Hamiltonian H has no self-duality $b \to b^{-1}$
2. 80% chance: Any reasonable definition of the model is massless for $b \geqslant 1$
3. 70% chance: Energy formula correct for $b \in [0, 1]$, and $e_0 = 0$ for $b \geqslant 1$.
4. 50% chance: FLZZ formula correct for *all* $a \geqslant (b + b^{-1})/2$
5. 50% chance: The model makes sense, without renormalization, for $b \geqslant 1$

Open problems:

- Rigorously construct the model for $b \geqslant 1/\sqrt{2}$
- Find if the model has an entanglement phase transition or crossover near $b = 1/\sqrt{2}$

Todo-list for continuous tensor networks

In $1+1$ dimensions

- Solve Fermion / Gauge theories
- Go into the $b \geqslant 1/\sqrt{2}$ of Sine-Gordon
- Do general CFT perturbations
- Compute more observables (masses, spectra, c-function...)

Todo-list for continuous tensor networks

In $1+1$ dimensions

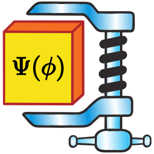
- Solve Fermion / Gauge theories
- Go into the $b \geqslant 1/\sqrt{2}$ of Sine-Gordon
- Do general CFT perturbations
- Compute more observables (masses, spectra, c-function...)

And of course the grand goal: do higher dimensions!

Come work on it in Paris!

Summary

$$|Q, R\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q \otimes \mathbb{1} + R \otimes a^\dagger(x)\right]\right\}|0\rangle_a$$



1. Ansatz for $1+1$ relativistic QFT
2. No cutoff, UV or IR, extensive, computable
3. Efficient (cost poly D, error plausibly 1/superpoly D)
4. **New:** larger D and reliable mass estimates
5. Works well for ϕ^4_2, Sine-Gordon, and Sinh-Gordon at $b \leqslant 1/\sqrt{2}$