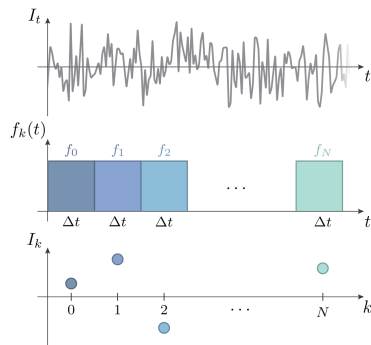


Digitized continuous quantum trajectories

continuous monitoring in the real world



Antoine Tilloy

January 31th, 2025

Quantum Trajectories

ICTS-TIFR Bangalore



The team



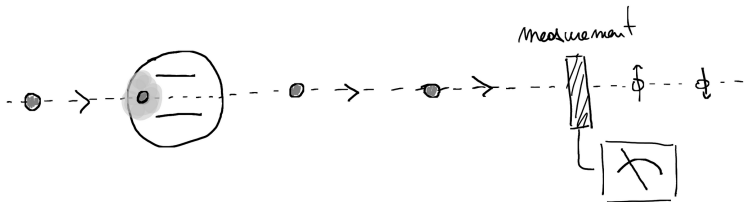
ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE



ALICE & BOB

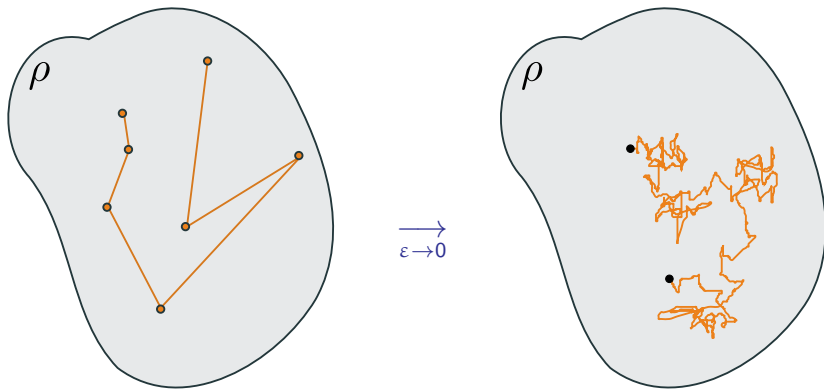
From discrete to continuous measurement

Continuous measurement – without Zeno effect



- ▶ time between ancillas $\Delta t \propto \epsilon$
- ▶ interaction strength $\omega \propto \sqrt{\epsilon}$

From discrete to continuous measurement



Continuous measurement (diffusive / homodyne case)

For single measured operator L we have the SME:

$$d\rho_t = \underbrace{\mathcal{L}(\rho_t) dt}_{\text{Lindblad}} + \underbrace{\mathcal{M}(\rho_t, dY_t)}_{\text{Measurement}}$$

with the signal $I_t = \frac{dY_t}{dt}$

$$dY_t = \sqrt{\eta} \text{tr}[(L + L^\dagger)\rho_t] dt + \underbrace{dW_t}_{\text{Wiener}}.$$

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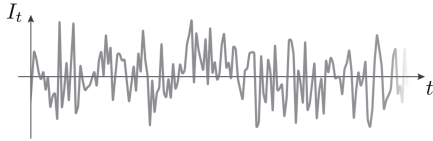
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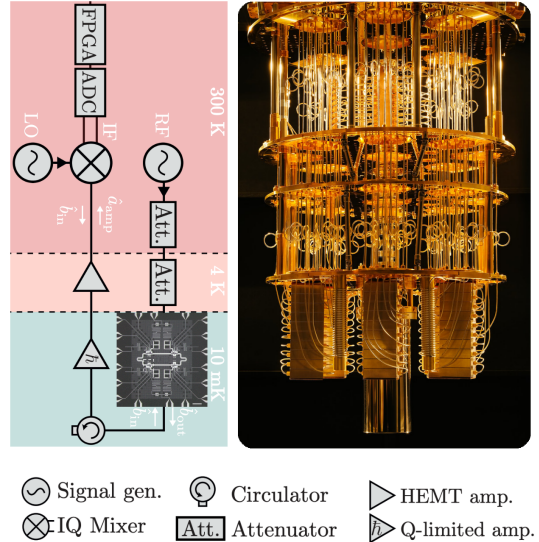
- today only homodyne – but applies to jump and heterodyne too

Continuous measurement signals are digitized



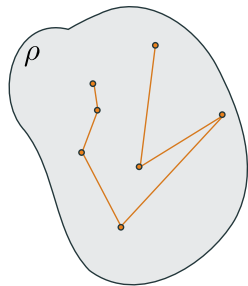
The *sharp* signal I_t is not empirically accessible:

- ▶ Formally it has the regularity of white noise (distribution)
- ▶ Practically, one can only store its average on time bins



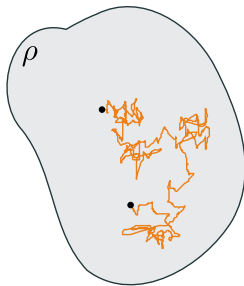
Blais et al. "Circuit quantum electrodynamics." Rev. Mod. Phys. (2021) – quantum computer image from IBM

From the discrete to the continuum and back



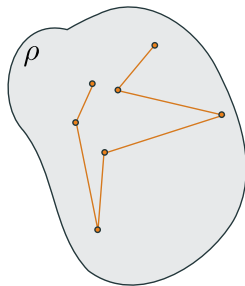
discrete time

$\xrightarrow{\varepsilon \rightarrow 0}$



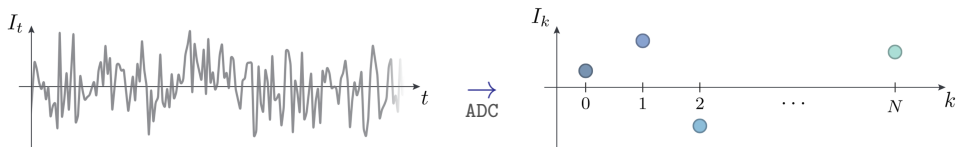
continuous time

$\xrightarrow{\text{digitization}}$



discretized continuous time

A problem in theory and practice

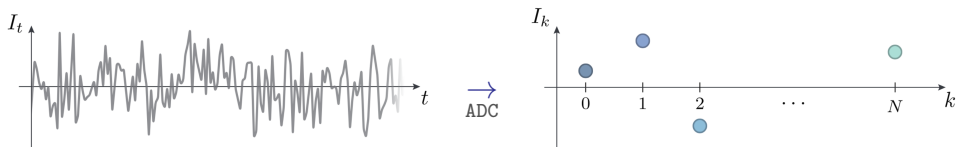


Currently: reconstruct with Euler, Runge-Kutta, or Rouchon scheme and

$$\frac{dY_t}{dt} \simeq \frac{\Delta Y_t}{\Delta t} = \frac{I_k}{\Delta t} \quad \text{with} \quad I_k = \int_{(k-1)\Delta t}^{k\Delta t} dY_t$$

push ADC to the max so $\Delta t \ll$ relevant timescales

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Problems:

- ▶ As we probe faster timescales, Δt is not so small...
- ▶ Δt small $\implies$ lots of data: at 1GHz in Float16, 4GB/s per quadrature
- ▶ Theoretically, we would like to know the error!

2 Solutions

Tasks like parameter estimation and max-like tomography are sensitive to Δt

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Possible fix:

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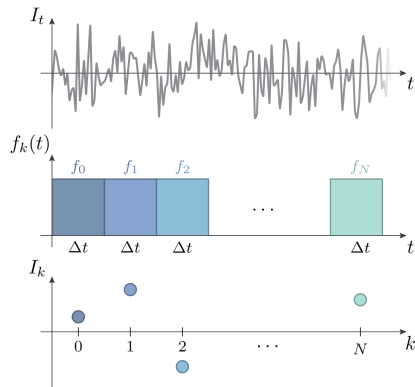
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2. Construct the Kraus map $\Phi_{\Delta t}^{I_k}(\rho)$ for a finite time bin Δt

Setup:



All we know is the binned signal

$$I_k = \int_{(k-1)\Delta t}^{k\Delta t} dY_t$$

Some information is gone: we cannot know the true ρ_t beyond order Δt !

Definition

The “binned” conditional state $\bar{\rho}_k$

$$\bar{\rho}_k := \mathbb{E} \left[\begin{array}{c|c} \rho_{k\Delta t} & l_k, l_{k-1}, \dots, l_1 \\ \text{true state} & \text{digitized signal} \end{array} \right]$$

- Clearly the best one can hope for → Bayesian optimum

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- ▶ In French we call it **robinet** (faucet) [binned ρ = rho binné = robinet]

Tool 1: Bayes' rule

$$\bar{\rho}_k = \mathbb{E} \left[\rho_{k\Delta t} \mid I_k, I_{k-1}, \dots, I_1 \right]$$

$$= \mathbb{E} \left[\rho_{k\Delta t} \mid I_k, \bar{\rho}_{k-1} \right]$$

$$= \frac{\mathbb{E} \left[\delta \left(I_k - \int_{k\Delta t}^{(k+1)\Delta t} dY_t \right) \rho_{k\Delta t} \mid \bar{\rho}_{k-1} \right]}{\mathbb{E} \left[\delta \left(I_k - \int_{(k-1)\Delta t}^{k\Delta t} dY_t \right) \mid \bar{\rho}_{k-1} \right]}$$

Definition

Markov

Bayes

Tool 2: Fourier transform

Write the Dirac δ in Fourier:

$$\delta(x) = \frac{1}{2\pi} \int_{\mathbb{R}} dp e^{ipx}$$

which implies

$$\delta\left(I_k - \int_{(k-1)\Delta t}^{k\Delta t} dY_t\right) = \frac{1}{2\pi} \int_{\mathbb{R}} dp \exp\left[ip\left(I_k - \int_{(k-1)\Delta t}^{k\Delta t} dY_t\right)\right]$$

Tool 3: Tilted Lindbladian

Define the p -tilted ρ :

$$\rho_t^{(p)} = \mathbb{E} \left[\exp \left(-ip \int_0^t dY_t \right) \rho_t \right]$$

Using Itô's lemma, one can show it obeys the p -tilted Lindblad equation

$$\frac{d\rho_t^{(p)}}{dt} = \mathcal{L} \cdot \rho_t^{(p)} - ip\mathcal{C}_L \cdot \rho_t^{(p)} - \frac{p^2}{2}\rho_t^{(p)}$$

with $\mathcal{C}_L \cdot \rho = \sqrt{\eta} (L\rho + \rho L^\dagger)$

Final formula

Putting all together:

Exact Kraus map

$$\tilde{\rho}_k = \frac{1}{2\pi} \int_{\mathbb{R}} dp \, e^{ip l_k - \Delta t \frac{p^2}{2}} \mathcal{T} \exp \left(\int_{(k-1)\Delta t}^{k\Delta t} \mathcal{L}_t - i p \mathcal{C}_L \right) \cdot \bar{\rho}_{k-1}$$

$$\bar{\rho}_k = \frac{\tilde{\rho}_k}{\text{tr}[\tilde{\rho}_k]}$$

- $\mathcal{T} \exp$ = time-ordered exponential = solution to linear ODE
(regular exponential if $\mathcal{L}$ time-independent)

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- ▶ $\mathcal{T} \exp$ = time-ordered exponential = solution to linear ODE (regular exponential if $\mathcal{L}$ time-independent)
- ▶ Is it numerically tractable?

Large time bins with Gaussian quadratures

Numerical integration (Folklore)

$1d$ integrals of smooth functions **numerically exact** with ≈ 100 points

$$\int f(x) dx \underset{\text{Float64}}{=} \sum_{j=1}^{100\text{-ish}} w_j f(x_j) \quad [\text{Gauss quadrature}]$$

Large time bins with Gaussian quadratures

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We include $\exp(-\Delta t \frac{p^2}{2})$ into the measure and use a Gauss-Hermite quadrature:

$$\begin{aligned} \tilde{\rho}_k &= \frac{1}{2\pi} \int_{\mathbb{R}} dp \, e^{ip l_k - \Delta t \frac{p^2}{2}} \mathcal{T} \exp \left(\int_{(k-1)\Delta t}^{k\Delta t} \mathcal{L} - i p \mathcal{C}_L \right) \cdot \bar{\rho}_{k-1} \\ &\underset{\text{Float64}}{=} \frac{1}{2\pi} \sum_j w_j e^{ip_j l_k} \underbrace{\mathcal{T} \exp \left(\int_{(k-1)\Delta t}^{k\Delta t} \mathcal{L} - i p_j \mathcal{C}_L \right)}_{\text{solution of linear ODE}} \cdot \bar{\rho}_{k-1} \end{aligned}$$

$\Rightarrow$ Solving (tilted) Lindblad equation on a time bin Δt for ≈ 100 different p_j

A first sanity check: signal probability for qubit

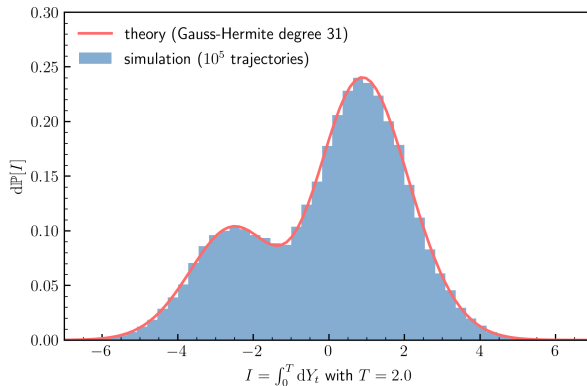
Empirical average with 10^6 very fine grained trajectories against analytical formula for 1 large time bin $\Delta t = T = 2$

$$H = \frac{1}{2}\sigma_x + \frac{1}{2}\sigma_y$$

$$L = \sigma_m$$

measure L with $\eta = 1$

$$\rho_0 = |e\rangle\langle e|$$



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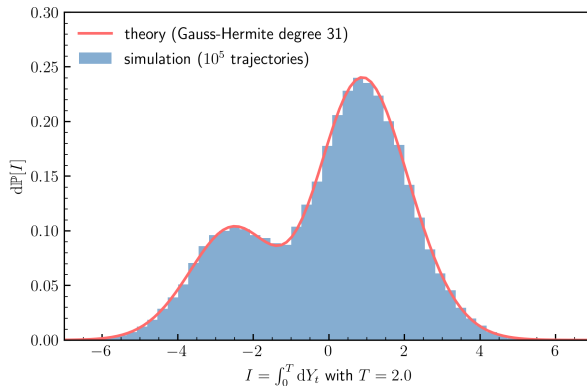
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► with MC: Qutip 10min $\rightarrow$ Dynamiqs CPU 20s $\rightarrow$ Dynamiqs GPU 0.7s

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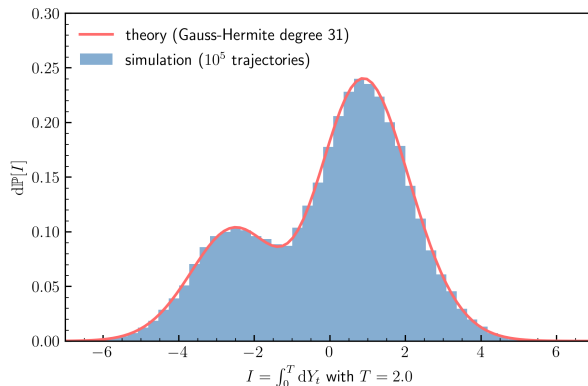
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- ▶ with MC: Qutip 10min $\rightarrow$ Dynamiqs CPU 20s $\rightarrow$ Dynamiqs GPU 0.7s
- ▶ with Robinet and 32 quadrature points: 0.5ms

Deflating a coherent state with 2 photon loss

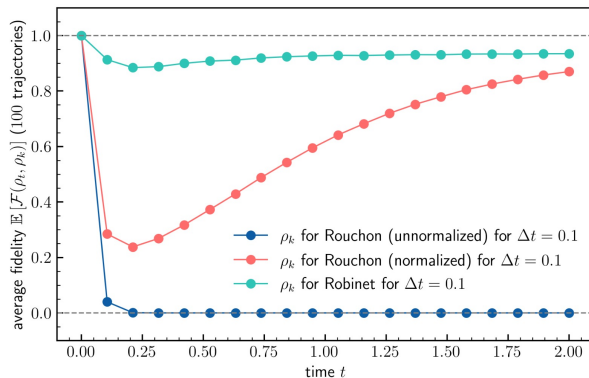
Simulate quantum trajectories with Robinet, Rouchon, Euler with coarse time bin $\Delta t = 0.1$ and compare fidelity with “true” state (simulated with $\delta t = 0.001$)

$$H = 0$$

$$L = a^2$$

measure L with $\eta = 0.5$

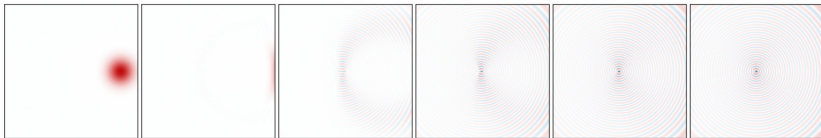
$$\rho_0 = |\alpha\rangle\langle\alpha| \text{ with } \alpha = 3$$



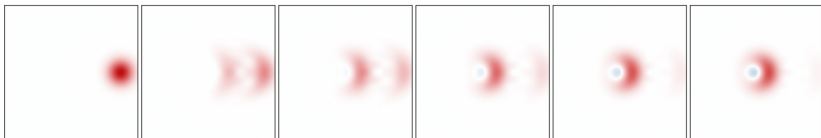
Deflating a coherent state with 2 photon loss

Averaged dynamics, qualitatively

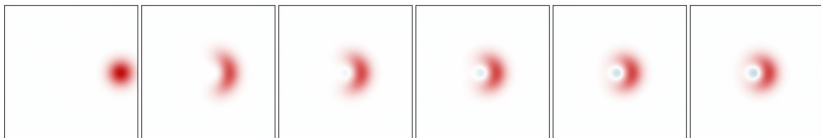
Euler



Rouchon



Robinet



True



A perturbative expansion

What are the corrections to standard discretizations?

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If Δt is small:

- ▶ $I_k \sim \sqrt{\Delta t} \rightarrow$ define $\mathcal{I}_k = I_k / \sqrt{\Delta t}$
- ▶ In the integral $p \sim 1/\sqrt{\Delta t} \rightarrow$ define $u = \sqrt{\Delta t} p$

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Assuming time-independent $\mathcal{L}$:

$$\tilde{\rho}_k = \frac{1}{2\pi\sqrt{\Delta t}} \int_{\mathbb{R}} du \, e^{iu\mathcal{I}_k - \frac{u^2}{2}} \underbrace{\exp\left(\Delta t \mathcal{L} - i\sqrt{\Delta t} u \mathcal{C}_L\right)}_{\text{Taylorable}} \cdot \bar{\rho}_{k-1}$$

can be expanded in $(\sqrt{\Delta t})^n$

Perturbative expansion continued

$$\begin{aligned}\tilde{\rho}_k \simeq \int_{\mathbb{R}} \frac{du e^{iu\mathcal{J}_k - \frac{u^2}{2}}}{2\pi\sqrt{\Delta t}} & \left(1 - \Delta t^{1/2} [i u \mathcal{C}_L] \right. \\ & + \Delta t \left[\mathcal{L} - \frac{u^2}{2} \mathcal{C}_L^2 \right] \\ & + \Delta t^{3/2} \left[\frac{iu^3}{6} \mathcal{C}_L^3 - \frac{iu}{2} (\mathcal{C}_L \mathcal{L} + \mathcal{L} \mathcal{C}_L) \right] \\ & \left. + \Delta t^2 \left[\frac{u^4}{24} \mathcal{C}_L^4 + \frac{-u^2}{6} (\mathcal{C}_L^2 \mathcal{L} + \mathcal{C}_L \mathcal{L} \mathcal{C}_L + \mathcal{L} \mathcal{C}_L^2) + \frac{1}{2} \mathcal{L}^2 \right] \right) \bar{\rho}_{k-1}\end{aligned}$$

The integrals in u are Gaussian!

Perturbative expansion continued

Exact filter expanded to order $\sqrt{\Delta t}^4 = \Delta t^2$:

$$\begin{aligned}\tilde{\rho}_k \simeq & \frac{e^{-\mathcal{J}_k^2/2}}{\sqrt{2\pi\Delta t}} \left(\mathbb{1} \right. \\ & + \sqrt{\Delta t}^1 [\mathcal{J}_k \mathcal{C}_L] \\ & + \sqrt{\Delta t}^2 \left[\mathcal{L} - \frac{(1 - \mathcal{J}_k^2)}{2} \mathcal{C}_L^2 \right] \\ & + \sqrt{\Delta t}^3 \left[\frac{\mathcal{J}_k(\mathcal{J}_k^2 - 3)}{6} \mathcal{C}_L^3 + \frac{\mathcal{J}_k}{2} (\mathcal{C}_L \mathcal{L} + \mathcal{L} \mathcal{C}_L) \right] \\ & \left. + \sqrt{\Delta t}^4 \left[\frac{(\mathcal{J}_k^4 - 6\mathcal{J}_k^2 + 3)}{24} \mathcal{C}_L^4 + \frac{(\mathcal{J}_k^2 - 1)}{6} (\mathcal{C}_L^2 \mathcal{L} + \mathcal{C}_L \mathcal{L} \mathcal{C}_L + \mathcal{L} \mathcal{C}_L^2) + \frac{1}{2} \mathcal{L}^2 \right] \right) \bar{\rho}_{k-1}\end{aligned}$$

Sampling trajectories

So far we assumed $l_1, \dots, l_n$ experimentally given $\rightarrow$ reconstruction problem

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where A_ℓ are simple traces of operators applied to $\bar{\rho}_{k-1}$

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Question: Does the scheme fit the criteria of Wonglakhon, Wiseman, and Chantasri 2408.14105?

Experimental tests of quantum trajectories

How to test quantum trajectory theory
[Benjamin Huard's talk]:

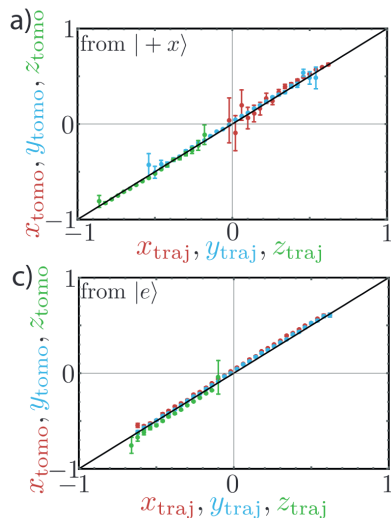
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⇒ powerful consistency check



Campagne-Ibarcq et al. PRX 2016

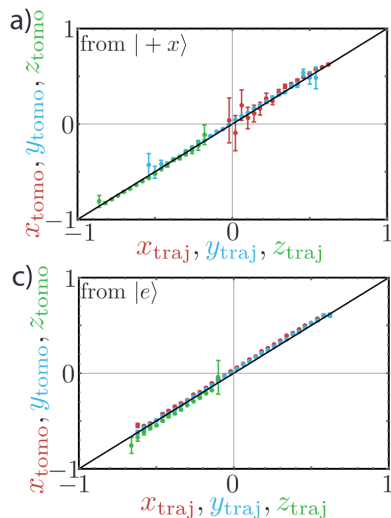
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Cannot in principle falsify poor use of signal
(e.g. mix with noise, and use lower efficiency in SME)



Campagne-Ibarcq et al. PRX 2016

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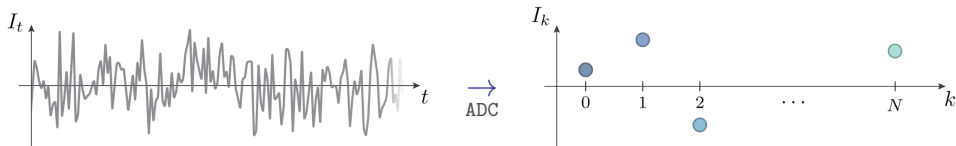
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Any suboptimal use of the signal will give lower fidelity

Summary



1. Real signals are digitized / discretized / binned
2. The corresponding Kraus map expressible exactly $\rightarrow$ Robinet
3. It can be computed numerically exactly with Gaussian quadratures
4. Or expanded perturbatively, giving systematic corrections of order $\Delta t^{n/2}$ to standard schemes
5. It can be used to reconstruct state from data or direct simulation
6. It always gives physically sound states
7. It makes quantum trajectory theory testable in a slightly stronger sense